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Abstract

Let $T$ be a Hamiltonian multipartite tournament with $n$ vertices and $\gamma$ a Hamiltonian cycle of $T$. We prove that for every $k$, $3 \leq k \leq \frac{5n}{2} + 4$, there exists a cycle $C$ of length $l(C) \in \{k - 3, k - 2, k - 1, k\}$, whose intersection with the arcs of $\gamma$ is at least $l(C) - 3$, and that the result is best possible.

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1 Introduction

The subject of pancyclicity in tournaments has been studied by several authors (e.g. [1],[4]). Two types of pancyclicity have been considered. A tournament $T$ is vertex-pancyclic if given any vertex $v$ there are cycles of every length containing $v$. Similarly, a tournament $T$ is arc-pancyclic if given any arc $e$ there are cycles of every length containing $e$. It is well known that a Hamiltonian tournament is vertex-pancyclic, but not necessarily arc-pancyclic. In a previous paper [8] we introduced the concept of cycle-pancyclicity to study questions such as the following. Given a cycle $C$, what is the maximum number of arcs which a cycle of length $k$ contained in the tournament induced by the vertices of $C$, $T[V(C)]$, has in common with $C$? Clearly, to study this kind of question it is sufficient to consider a Hamiltonian tournament where $C$ is a Hamiltonian cycle of $T$.

In this paper we consider a Hamiltonian multi-partite tournament $T$ with vertex set $V = \{0, 1, \ldots, n - 1\}$ and arc set $A$. And we will assume without loss of generality that $\gamma = \ldots$
(0,1,\ldots,n−1,0) is a Hamiltonian cycle of \(T\). Let \(C_k\) denote a directed cycle of length \(k\). For a cycle \(C_k\) we denote \(\mathcal{I}(C_k) = |A(\gamma) \cap A(C_k)|\), or simply \(\mathcal{I}(C_k)\) when \(\gamma\) is understood. Let \(f(n,k,T) = \max\{\mathcal{I}(C_k)|C_k \subseteq T\}\) and \(f(n,k) = \min\{f(n,k,T)|T\text{ is a Hamiltonian tournament with } n \text{ vertices}\}\). In [8, 9, 10] we proved that \(f(n,k)\) is either \(k−3\) or \(k−4\), depending on weather \(n ≥ 2k−4\) or \(n < 2k−4\), respectively.

It is well known that a Hamiltonian bipartite tournament is pancyclic, and vertex-pancyclic (with only very few exceptions) but not necessarily arc-pancyclic (see e.g. [4, 13, 16]). Cycle-pancyclicity has also been studied in bipartite tournaments. In [6] it was proved that for even \(k\) in the range \(4 ≤ k ≤ \frac{n+4}{2}\), there exist a cycle \(C\) of length \(l(C) \in \{k−2,k\}\) intersecting \(\gamma\) in at least \(l(C)−3\) arcs. For the case of \(k > \frac{n+4}{2}\), it was proved in [7] that the intersection is at least \(l(C)−4\) arcs.

The study of pancyclicity in multipartite tournaments was initiated by Bondy in 1976 [2]. Multipartite tournaments satisfy a restricted type of vertex-pancyclicity: It is known that every partite set of a strongly connected \(p\)-partite tournament has at least one vertex that lies on cycles of each length \(m\) for \(m \in \{3,4,\ldots,p\}\) [12]. We know of no similar results for cycles of length greater than \(p\), and we know of no pancyclicity results for Hamiltonian multipartite tournaments. For two recent extensive surveys on cycles in multipartite tournaments see [5, 15]. In [15] the following two open problems are mentioned, from [14].

- Conjecture 2.31. Let \(D\) be a regular \(p\)-partite tournament with \(p ≥ 5\). Then \(D\) contains a strongly connected sub-tournament of order \(p\).

- Problem 2.32. Determine further sufficient condition for (strongly connected) \(p\)-partite tournaments to contain a strong sub-tournament of order \(c\) for some \(4 ≤ c ≤ n\).

In this paper we prove that, for every \(k, 4 ≤ k ≤ \frac{n+4}{2}\), for every Hamiltonian multipartite tournament, there exists a cycle \(C\) of length \(l(C) \in \{k−3,k−2,k−1,k\}\), whose intersection with the arcs of \(\gamma\) is at least \(l(C)−3\).

As a consequence of our result, we obtain an affirmative answer to Conjecture 2.31 in case that \(D\) is Hamiltonian and not necessarily regular. Another consequence of this result is similar to Problem 2.32:

- Let \(T\) be a Hamiltonian multipartite tournament of order \(n\), and \(4 ≤ k ≤ \frac{n+4}{2}\). Then \(T\) contains a Hamiltonian sub-tournament of order \(v\) with \(v \in \{k−3,k−2,k−1,k\}\).

The rest of this paper is organized as follows. In Section 2 some notation and basic results needed in the rest of the paper are introduced. The proof of the main result, i.e., that there exists a cycle \(C_{h(k)}\), \(h(k) \in \{k−3,k−2,k−1,k\}\) such that \(\mathcal{I}(C_{h(k)}) ≥ h(k)−3\), for \(n ≥ 2k−4\) appears in Section 3, 4, 5 and 6. Sections 3 through 5 contain special cases (for particular values of \(n\) and \(k\)). The general case is left to Section 6.
2 Preliminaries

In all of this paper all notation is taken modulo $n$. Recall that in this paper we consider a Hamiltonian multi-partite tournament $T$ with vertex set $V = \{0, 1, \ldots, n-1\}$ and arc set $A$, and $\gamma = (0, 1, \ldots, n-1, 0)$ is an arbitrary Hamiltonian cycle of $T$. Vertex $u$ is adjacent to vertex $v$ iff either $(u, v) \in A(T)$ or $(v, u) \in A(T)$. We denote by $C_i$ a cycle of length $l$. For a cycle $C$, let $I(C) = |A(C) \cap A(\gamma)|$.

2.1 Chords and Quasichords

A chord of a cycle $C$ is an arc not in $C$ with both terminal vertices in $C$. The length of a chord $f = (u, v)$ of $C$, denoted $l(f)$, is equal to the length of $\langle u, C, v \rangle$, where $\langle u, C, v \rangle$ denotes the $uv$-directed path contained in $C$. We say that $f$ is a $c$-chord if $l(f) = c$ and $f = (u, v)$ is a $-c$-chord if $l(v, C, u) = c$. Observe that if $f$ is a $c$-chord then it is also a $-(n-c)$-chord. For the following definition, notice that if $u$ is not adjacent to $v$ then $u$ is adjacent to $v - 1$ and $u$ is adjacent to $v + 1$, because $T$ is a multi-partite tournament.

Definition 2.1 (Quasichord) An ordered pair $(u, v)$ is a quasichord of $\gamma$ if the following properties are satisfied:

1. if $u$ is adjacent to $v$ in $T$ then $(u, v) \in A(T)$
2. if $u$ is not adjacent to $v$ in $T$ then $\{(u, v - 1), (u, v + 1)\} \subseteq A(T)$.

If $l(u, v, v) = c$ we say that $f$ is a $c$-quasichord, and that it is a $-(n-c)$-quasichord.

Observe that every chord of $\gamma$ is also a quasichord of $\gamma$, and that it may be that for a pair of vertices $u, v$, both $(u, v)$ and $(v, u)$ are quasichords of $\gamma$.

Remark 1 If $(u, v)$ is not a quasichord then at least one of the following three chords is in $A$: $(v, u)$, $(v - 1, u)$, $(v + 1, u)$.

Definition 2.2 (Quasicycle) An $s$-quasicycle $Q_m$ is a succession $u_0, u_1, \ldots, u_{m-1}, u_0$ of distinct vertices such that $m \geq 3$ and there exists $\{i_0, i_1, \ldots, i_s\} \subseteq \{0, 1, \ldots, m - 1\}$ that satisfies

1. for each $i \in \{\{0, 1, \ldots, m - 1\} - \{i_0, i_1, \ldots, i_s\}\}$, the arc $(u_i, u_{i+1})$ is in $A(\gamma)$.
2. for each $j \in \{i_0, i_1, \ldots, i_s\}$, the pair $(u_j, u_{j+1})$ is a quasichord of $\gamma$.

We say that the length of the quasicycle $Q_m$ is $l(Q) = m$. 

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Definition 2.3 Let $B$ be a set of chords of $\gamma$. We say that $B$ induces a cycle of $T$ with respect to $\gamma$, if there exists a cycle $C$, $A(C) \subseteq A(\gamma) \cup B$ such that $B \subseteq A(C)$. And we say that $C$ is the cycle of $T$ induced by $B$ with respect to $\gamma$, and denote it $\gamma[B]$.

Notice that if such a cycle exists then it is unique.

Definition 2.4 If $B \subseteq A(T)$ induces a cycle with respect to $\gamma$ then we will consider the following order for $B$; see Figure 1. Let $(u_i, v_i)$ be any arc of $B$. Once $(u_i, v_i)$ is defined, we denote by $(u_{i+1}, v_{i+1})$ the arc of $B$ such that $u_{i+1}$ is the first vertex of $C$ (in the order implied by $C$) after $u_i$ that is the initial vertex of some arc of $B$. Thus we have the set $\{(u_i, v_i) : i \in \{1, 2, \ldots, n\}\}$. We denote by $T_i = (v_i, \gamma, u_i+1) \subseteq \gamma$. Also, we denote by $T'_i = (z_i, \gamma, v_i)$ where $z_i$ is the first vertex of the cycle induced by $B$ after $v_i$ traversing $\gamma$ in the opposite direction. Finally, $l_i = l(T_i)$ and $l'_i = l(T'_i)$.

Lemma 2.1 If there exists a 1-quasicycle $Q_{l+1}$ in $T$, then at least one of the following two conditions holds:

1. there exists a cycle $C_{l+1}$ with $I(C_{l+1}) = l$;
2. there exists a cycle $C_{l+2}$ with $I(C_{l+2}) = l+1$, and there exists a cycle $C_l$ with $I(C_l) = l-1$.

Proof: Let $Q_{l+1} = (0, 1, \ldots, l, 0)$ be a 1-quasicycle with $(l, 0)$ a quasichord of $\gamma$. When $l$ is adjacent to 0 we have that $(l, 0) \in A(T)$, and then it suffices to take $C_{l+1} = \gamma[(l, 0)]$. When $l$ is not adjacent to 0 we have that $\{(l, n-1), (l, 1)\} \subseteq A(T)$. We take $C_{l+2} = \gamma[(l, n-1)]$ and $C_l = \gamma[(l, 1)]$.

Lemma 2.2 If there exists a 2-quasicycle $Q_{l+2}$ in $T$, then there exists a cycle $C_{h(l)}$ with $I(C_{h(l)}) \geq h(l) - 2$, $h(l) \in \{l+2, l+3\}$, $l = l_1 + l_2$, where $l_1 = l(T_1)$, and $l_2 = l(T_2)$.

Proof: Let $Q_{l+2} = (0, x, x+1, \ldots, x + l_1 = y, z, z+1, \ldots, z + l_2 = n = 0)$ be a 2-quasicycle with $(0, x)$ and $(y, z)$ quasichords.
Case 1: \( l_1 \geq 1 \) and \( l_2 \geq 1 \).
If \( \{(0, x), (y, z)\} \subseteq A(T) \) then let \( C_{l+2} = \gamma[\{(0, x), (y, z)\}] \).
If \( (0, x) \in A(T) \) and \( (y, z) \notin A(T) \) then let \( C_{l+3} = \gamma[\{(0, x), (y, z-1)\}] \).
If \( (0, x) \notin A(T) \) and \( (y, z) \in A(t) \) then let \( C_{l+3} = \gamma[\{(0, x-1), (y, z)\}] \).
If \( (0, x) \notin A(T) \) and \( (y, z) \notin A(t) \) then let \( C_{l+2} = \gamma[\{(0, x+1), (y, z-1)\}] \).

Case 2: \( l_1 \geq 1 \) and \( l_2 = 0 \). In this case \( z = 0 \).
If \( \{(0, x), (y, 0)\} \subseteq A(T) \) then let \( C_{l+2} = \gamma[\{(0, x), (y, 0)\}] \).
If \( (0, x) \in A(T) \) and \( (y, 0) \notin A(T) \) then let \( C_{l+3} = \gamma[\{(0, x), (y, n-1)\}] \).
If \( (0, x) \notin A(T) \) and \( (y, 0) \in A(t) \) then let \( C_{l+3} = \gamma[\{(0, x-1), (y, 0)\}] \).
If \( (0, x) \notin A(T) \) and \( (y, 0) \notin A(t) \) then let \( C_{l+2} = \gamma[\{(0, x-1), (y, z+1)\}] \).

Case 3: \( l_1 = 0 \) and \( l_2 \geq 1 \). This is analogous to Case 2.

Case 4: \( l_1 = 0 \) and \( l_2 = 0 \). This case is impossible due to the definition of quasicyclic.

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**Lemma 2.3** Assume there exists a 3-quasicycle \( Q_{l+3} \) in \( T \) with \( l + 3 \geq 4 \), \( l = l_1 + l_2 + l_3 \), such that for each \( i \in \{1, 2, 3\} \) if \( l(T_i) = 0 \) then \( l(T'_i) \geq 2 \). Then there exists a cycle \( C_{h(l)} \) with \( h(l) \in \{l + 3, l + 3 + \Delta, l + 3 + 2\Delta\} \), for a fixed \( \Delta, \Delta \in \{-1, +1\} \), and with \( \mathcal{I}(C_{h(l)}) \geq h(l) - 3 \).

**Proof:** Let \( f_i = (u_i, v_i) \) with \( i \in \{1, 2, 3\} \) the three quasischord of \( Q_{l+3} \).

**Case 1:** \( f_i \in A(T) \) for all \( i \in \{1, 2, 3\} \).
Let \( C_{h(l)} = \gamma[\{(f_1, f_2, f_3)\}] \). Clearly \( h(l) = l + 3 \) and \( \mathcal{I}(C_{h(l)}) = l \).

**Case 2:** exactly one \( f_i \notin A(T) \), say \( f_1 \) is not in \( A(T) \) and \( \{f_2, f_3\} \subseteq A(T) \).
If \( l(T_1) \geq 1 \) then \( C_{h(l)} = \gamma[\{(u_1, v_1 + 1), f_2, f_3\}] \), clearly \( h(l) = l + 2, \Delta = -1 \) and \( \mathcal{I}(C_{h(l)}) = h(l) - 3 \).
If \( l(T_1) = 0 \) it follows from the lemma hypothesis that \( l(T'_1) \geq 2 \) and \( C_{h(l)} = \gamma[\{(u_1, v_1 - 1), f_2, f_3\}] \), clearly \( h(l) = l + 4, \Delta = +1 \) and \( \mathcal{I}(C_{h(l)}) = h(l) - 3 \).

**Case 3:** there exist two quasischords in \( \{f_1, f_2, f_3\} \), say \( f_1, f_2 \), such that \( \{f_1, f_2\} \cap A(T) = \emptyset \), and \( f_3 \in A(T) \).
If \( l(T_1) \geq 0 \) and \( l(T_2) > 0 \) then let \( C_{h(l)} = \gamma[\{(u_1, v_1 + 1), (u_2, v_2 + 1), f_3\}] \). Clearly \( h(l) = l + 1 = l + 3 + 2\Delta, \Delta = -1 \) and \( \mathcal{I}(C_{h(l)}) = l - 2 \).
If \( l(T_1) > 0 \) and \( l(T_2) = 0 \) then let \( C_{h(l)} = \gamma[\{(u_1, v_1 + 1), (u_2, v_2 - 1), f_3\}] \). Clearly \( h(l) = l + 3 \) with \( \mathcal{I}(C_{h(l)}) = l \).
The case of \( l(T_1) = 0 \) and \( l(T_2) > 0 \) is analogous to the previous case.
If \( l(T_1) = 0 \) and \( l(T_2) = 0 \) then let \( C_{h(l)} = \gamma[\{(u_1, v_1 - 1), (u_2, v_2 - 1), f_3\}] \). Clearly \( h(l) = l + 5 = l + 3 + 2\Delta, \Delta = +1 \), and \( \mathcal{I}(C_{h(l)}) = l + 2 \).
Case 4: $f_1, f_2, f_3 \notin A(T)$. Since $4 \leq l(Q_{l+3}) = l + 3 < n - 3$ there exist $i, j \in \{1, 2, 3\}$ such that $l(T_i) > 0$ and $l(T_j) > 1$.

If $i = j$ then there exists $k \in \{1, 2, 3\} - \{i\}$ such that $l(T_k) > 0$. Let

$$f'_s = \begin{cases} (u_s, v_s + 1) & \text{if } l(T_s) > 0 \\ (u_s, v_s - 1) & \text{if } l(T_s) = 0 \end{cases}$$

where $s \in \{1, 2, 3\} - \{i, k\}$. Let $C_{h(l)} = \gamma[\{(u_k, v_k + 1), (u_i, v_i - 1), f'_s\}]$. Clearly $h(l) \in \{l + 2, l + 4\}$; if $l(T_s) > 0$ then $\Delta = -1$; if $l(T_s) = 0$ then $\Delta = +1$. Also $\mathcal{I}(C_{h(l)}) \geq h(l) - 3$.

If $i \neq j$ let $k \in \{1, 2, 3\} - \{i, j\}$, and

$$f'_k = \begin{cases} (u_k, v_k + 1) & \text{if } l(T_k) > 0 \\ (u_k, v_k - 1) & \text{if } l(T_k) = 0 \end{cases}$$

Let $C_{h(l)} = \gamma[\{(u_i, v_i + 1), (u_j, v_j - 1), f'_k\}]$. Clearly $h(l) \in \{l + 2, l + 4\}$; if $l(T_k) > 0$ then $\Delta = -1$; if $l(T_k) = 0$ then $\Delta = +1$. Also $\mathcal{I}(C_{h(l)}) \geq h(l) - 3$.

Notice that in the previous lemma $h(l) = l + 5$ only in the case that two of the quasichords, say $f_1, f_2$ are not in $A(T)$, and $l(T_1) = l(T_2) = 0$.

2.2 Basic Properties

For any $a$, $2 \leq a \leq n - 2$, denote by $t_a$ the largest integer such that $a + t_a(k - 3) < n - 1$, where $n$ is the number of vertices of $T$. The important case of $t_{k-2}$ is denoted by $t$ in the rest of the paper. Let $r$ be defined as follows: $r = n - [k - 2 + t(k - 3)]$.

Notice the following facts.

- If $a \leq b$, then $t_a \geq t_b$.
- $t \geq 0$.
- $2 \leq r \leq k - 2$.
- $n = k - 2 + t(k - 3) + r$.

Lemma 2.4 If the $a$-quasichord with initial vertex 0 (recall that 0 is an arbitrary vertex of $T$) is in $A$, then at least one of the two following properties holds.

(i) Exists a cycle $C_{h(k)}$ of length $h(k) \in \{k - 2, k - 1, k\}$ with $\mathcal{I}(C_{h(k)}) \geq h(k) - 3$. 


(ii) For every $0 \leq i \leq t_o$, $(0,a + i(k - 3))$ is a quasichord.

Proof: Suppose that (ii) in the lemma is false, and let

$$j = \min\{i \in \{1, 2, \ldots, t_o\} \mid (0,a + i(k - 3)) \text{ is not a quasichord}\}.$$

Case 1: If $0$ and $a + j(k - 3)$ are adjacent, then $(a + j(k - 3), 0) \in A$.

$$C'_{k-1} = \langle a + (j-1)(k-3), \gamma, a + j(k-3) \rangle \cup \langle a + j(k-3), 0, a + (j-1)(k-3) \rangle$$

is a 2-quasicycle of length $k - 1$. By Lemma 2.2 there exists a cycle of length $h(k)$ with $h(k) \in \{k - 1, k\}$ such that $\mathcal{I}(C_{h(k)}) \geq h(k) - 3$.

Case 2: If $0$ and $a + j(k - 3)$ are not adjacent, then $0$ and $a + j(k - 3) + 1$ are adjacent, and also $0$ and $a + j(k - 3) - 1$ are adjacent. And since $(0,a + j(k - 3))$ is not a quasichord, there are two possibilities.

Case 2.1: If $(a+j(k-3)-1,0) \in A$ then

$$C'_{k-2} = \langle a + j(k-3) - 1, 0, a + (j-1)(k-3) \rangle \cup \langle a + (j-1)(k-3), \gamma, a + j(k-3) \rangle$$

is a 2-quasicycle of length $k - 2$. By Lemma 2.2 there exists a cycle of length $h(k)$ with $h(k) \in \{k - 2, k - 1\}$ such that $\mathcal{I}(C_{h(k)}) \geq h(k) - 2$.

Case 2.2: If $(a+j(k-3)+1,0) \in A$ then let $C' = \gamma[\{(a+j(k-3)+1,0),(0,a+(j-1)(k-3))\}]$ when $(0,a+(j-1)(k-3)) \in A$, and $C' = \gamma[\{(a+j(k-3)+1,0),(0,a+(j-1)(k-3)-1)\}]$ when $(0,a+(j-1)(k-3)) \notin A$. Clearly $\ell(C') \in \{k - 1, k\}$, and $\mathcal{I}(C') = \ell(C') - 2$.

The following is a consequence of Lemma 2.4.

**Corollary 2.5** At least one of the two following properties holds.

(i) Exists a cycle $C_{h(k)}$ of length $h(k)$ such that $\mathcal{I}(C_{h(k)}) \geq h(k) - 3$, and $h(k) \in \{k - 2, k - 1, k\}$.

(ii) For every $0 \leq i \leq t$, every $(0,(k-2) + i(k-3))$ is a quasichord.

Proof: We only have to show that $(0,k - 2)$ is a quasichord, and the result follows from Lemma 2.4.

Case 1: If $0$ and $k - 2$ are adjacent then $(0, k - 2) \in A$ since otherwise $\gamma[(k - 2, 0)]$ is a cycle $C_{k-1}$ with $\mathcal{I}(C_{k-1}) = k - 2$.

Case 2: If $0$ and $k - 2$ are not adjacent then we prove that $\{(0, k - 1), (0, k - 3)\} \subseteq A$. When $(k-1,0) \in A$ we have that $C' = \gamma[(k-1,0)]$ has length $k$ and $\mathcal{I}(C') = k - 1$. When $(k-3,0) \in A$, we have that $C' = \gamma[(k-3,0)]$ has length $k - 2$ and $\mathcal{I}(C') = k - 3$. □
3 The Cases $k = 4, 5$

Theorem 3.1 Exists a cycle $C_3$, $I(C_3) \geq 1$, or exists a cycle $C_4$, $I(C_4) \geq 2$.

Proof: Let $i = \min\{j \in V : (j, 0) \in A\}$. Observe that $i$ is well defined since $(n - 1, 0) \in A$. If $i - 1$ and $0$ are adjacent, then by the choice of $i$ we have $(0, i - 1) \in A$ and $C_3 = (0, i - 1, i, 0)$ is a cycle with $I(C_3) \geq 1$. If $i - 1$ and $0$ are not adjacent, then $0$ and $i - 2$ are adjacent, and by the choice of $i$, $(0, i - 2) \in A$, and $C_4 = (0, i - 2, i - 1, i, 0)$ is a cycle with $I(C_4) \geq 2$. ■

Theorem 3.2 Exists a cycle $C_4$, $I(C_4) \geq 1$, or exists a cycle $C_5$, $I(C_5) \geq 2$.

Proof: We consider the following cases.

$n \equiv 2 \pmod{3}$ Notice that $r_5 = n - 4$. Considering Lemma 2.4 we have that at least one of the following statements holds: $(0, n - 4) \in A$ or $\{(0, n - 5), (0, n - 3)\} \subseteq A$. If $(0, n - 4) \in A$ then $C_5 = (0, n - 4, n - 3, n - 2, n - 1, 0)$ satisfies $I(C_5) = 4$; otherwise $(0, n - 3) \in A$ and then $C_4 = (0, n - 3, n - 2, n - 1, 0)$ satisfies $I(C_4) = 3$.

$n \equiv 1 \pmod{3}$ Notice that $r_5 = n - 3$ and by Lemma 2.4 we have that $(0, n - 3) \in A$ or $\{(0, n - 2), (0, n - 4)\} \subseteq A$. If $(0, n - 3) \in A$ then $C_4 = (0, n - 3, n - 2, n - 1, 0)$ satisfies $I(C_4) = 3$; otherwise $(0, n - 4) \in A$ and then $C_5 = (0, n - 4, n - 3, n - 2, n - 1, 0)$ satisfies $I(C_5) = 4$.

$n \equiv 0 \pmod{3}$ Notice that $r_5 = n - 2$ and by Lemma 2.4 we have that $(0, n - 5) \in A$ or $\{(0, n - 4), (0, n - 6)\} \subseteq A$. If $(0, n - 5) \in A$ then $C_6 = (0, n - 5, n - 4, n - 3, n - 2, n - 1, 0)$ satisfies $I(C_6) = 5$; otherwise $(0, n - 4) \in A$ and then $C_5 = (0, n - 4, n - 3, n - 2, n - 1, 0)$ satisfies $I(C_5) = 4$. ■

4 The case of $n = 2k - 4$

Theorem 4.1 If $n = 2k - 4$ then exists a cycle $C_{h(k)}$, $h(k) \in \{k - 2, k - 1, k\}$ such that $I(C_{h(k)}) \geq h(k) - 1$.

Proof: Let $x$ and $y$ be two vertices of $T$ such that $l(x, y, y) = l(y, y, x) = k - 2$. Without loss of generality we can assume that $x = 0, y = k - 2$.

Case 1: 0 and $k - 2$ are adjacent. If $(0, k - 2) \in A$ then $C_{k-1} = (0, k - 2) \cup (k - 2, \gamma, 0)$ is a directed cycle with $I(C_{k-1}) = k - 2$. If $(k - 2, 0) \in A$ then $C_{k-1} = (0, \gamma, k - 2) \cup (k - 2, 0)$ is a directed cycle with $I(C_{k-1}) = k - 2$. 

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Case 2: 0 and $k - 2$ are not adjacent. It is clear that 0 and $k - 3$ are adjacent. If $(0, k - 3) \in A$ then $C_k = (0, k - 3) \cup (k - 3, \gamma, 0)$ is a directed cycle with $I(C_k) = k - 1$. If $(k - 3, 0) \in A$ then $C_{k-2} = (0, \gamma, k - 3) \cup (k - 3, 0)$ is a directed cycle with $I(C_{k-2}) = k - 3$.

5 The cases $r = k - 2$, $r = k - 3$ and $r = k - 4$

Theorem 5.1 If $r = k - 2$, $r = k - 3$, or $r = k - 4$ then exists a cycle $C_{h(k)}$, $h(k) \in \{k - 2, k - 1, k\}$ such that $I(C_{h(k)}) \geq h(k) - 3$.

Proof: By Corollary 2.5, taking $i = t$, at least one of the two following assertions is valid:

1. exists a cycle $C_{h(k)}$, $h(k) \in \{k - 2, k - 1, k\}$ such that $I(C_{h(k)}) \geq h(k) - 3$, or
2. $(0, (k - 2) + t(k - 3))$ is a quasichord.

If assertion 1 holds the theorem follows. Assume then that it does not hold.

For the case $r = k - 2$ (resp. $r = k - 3$, $r = k - 4$): when 0 and $k - 2 + t(k - 3)$ are adjacent we have that $(0, k - 2 + t(k - 3)) \in A$ and $(0, k - 2 + t(k - 3)) \cup (k - 2 + t(k - 3), \gamma, 0)$ is a cycle $C$ of length $k - 1$ (resp. $k - 2$, $k - 3$) with $I(C) \geq \ell(C) - 3$.

When 0 and $k - 2 + t(k - 3)$ are not adjacent we have that $(0, k - 2 + t(k - 3) - 1) \in A$ and $(0, k - 2 + t(k - 3) - 1) \cup (k - 2 + t(k - 3) - 1, \gamma, 0)$ is a cycle $C$ of length $k$ (resp. $k - 1$, $k - 2$) with $I(C) \geq \ell(C) - 3$.

Corollary 5.2 If $t = 0$ then exists a cycle $C_{h(k)}$, $h(k) \in \{k - 2, k - 1, k\}$ such that $I(C_{h(k)}) \geq h(k) - 3$.

Proof: If $t = 0$ then $n = k - 2 + r$, where $k - 2 \leq r \leq k - 2$ since $n \geq 2k - 4$. It follows that $r = k - 2$ and the result follows directly from Theorem 5.1.

6 The General Case

In this section we assume that $r \leq k - 5$, $k \geq 6$, $t \geq 1$, since the other cases have been considered in the previous sections.

The next lemma follows directly from Lemma 2.4.

Lemma 6.1 If the $(k - 2 + \alpha)$-quasichord, $-(k - 4) \leq \alpha \leq r$, with initial vertex 0 is in $A$, then at least one of the two following properties holds.
(i) There exists a cycle $C_{h(k)}$, $h(k) \in \{k - 2, k - 1, k\}$ such that $I(C_{h(k)}) \geq h(k) - 3$.

(ii) For every $0 \leq i \leq t - 1$, the $(k - 2 + \alpha + i(k - 3))$-quasichord with initial vertex 0 is in $A$.

**Lemma 6.2** At least one of the two following properties holds.

(i) There exists a cycle $C_{h(k)}$, $h(k) \in \{k - 3, k - 2, k - 1, k\}$ such that $I(C_{h(k)}) \geq h(k) - 3$.

(ii) All the following quasichords are in $A$.

(a) Every $(k - 2)$-quasichord.

(b) Every $(-r)$-quasichord.

(c) Every $-(r + 1)$-quasichord.

(d) Every $(k - 3)$-quasichord.

**Proof:** The proof of (a) follows directly from Corollary 2.5 taking $i = 0$.

The proof of (b) follows from Corollary 2.5 taking $i = t$ and recalling that $n - r = k - 2 + t(k - 3)$.

To prove (c) suppose that (c) does not hold. Then exist $y, x$, $x = y + r + 1$, such that $(y, x)$ is not a quasichord. It follows from Remark 1 that at least one of the following chords is in $A$: $(x, y), (x - 1, y), (x + 1, y)$.

Case (c.1): $g_1 = (x, y) \in A$. By (a) we can take $a = k - 2$ in Lemma 2.4; taking $i = t - 1$ in the same lemma, we can consider the $(k - 2 + (t - 1)(k - 3))$-quasichord, namely $f_1$, that starts in $y + k - 4$. Notice that $f_1$ ends in $x$. By Lemma 2.2, considering the 2-quasicycle generated by $\{g_1, f_1\}$, we have that there exists a cycle $C_{h(k)}$ with $h(k) \in \{k - 2, k - 1\}$ and $I(C_{h(k)}) \geq h(k) - 2$. (Notice that $\ell = k - 4$ in Lemma 2.2).

Case (c.2): $g_2 = (x + 1, y) \in A$. By Lemma 2.2, considering the 2-quasicycle generated by $\{g_2, f_1\}$, it follows that there exists a cycle $C_{h(k)}$ with $h(k) \in \{k - 1, k\}$ and $I(C_{h(k)}) \geq h(k) - 2$. (Notice that $\ell = k - 3$ in Lemma 2.2).

Case (c.3): $g_3 = (x - 1, y) \in A$. The $(k - 2 + (t - 1)(k - 3))$-quasichord that starts in $y + k - 5$ ends in $x - 1$. It follows from Lemma 2.2, considering the 2-quasicycle generated by $\{g_3, f_2\}$, that there exists a cycle $C_{h(k)}$ with $h(k) \in \{k - 3, k - 2\}$ and $I(C_{h(k)}) \geq h(k) - 2$.

Finally to prove (d) assume (d) does not hold, and we prove (i) of the lemma. Suppose without loss of generality that $(0, k - 3)$ is not a quasichord of $T$. Case (d.1) $x = 0$ is adjacent to $k - 3$. In this case, $(k - 3, 0) \in A$ and $C_{k - 2} = \gamma[[k - 3, 0]]$ has $I(C_{k - 2}) = k - 3$. Case (d.2) $x = 0$ is not adjacent to $k - 3$. It follows that $(k - 2, 0) \in A$ or $(k - 4, 0) \in A$. If $(k - 2, 0) \in A$ then $C_{k - 3} = \gamma\{(k - 2, 0)\}$ has $I(C_{k - 3}) = k - 4$. If $(k - 4, 0) \in A$ then $C_{k - 3} = \gamma\{(k - 4, 0)\}$ has $I(C_{k - 3}) = k - 4$. \[\]
Lemma 6.3 Let \(-1 \leq i \leq r\). If all the \(-r\)-quasichords, \(-(r+1)\)-quasichords, \((k-3+i)\)-quasichords and \((k-2+i)\)-quasichords are in \(T\) then at least one of the following properties holds.

(i) There exists a cycle \(C_{h(k)}\), \(h(k) \in \{k-3, k-2, k-1, k\}\) such that \(I(C_{h(k)}) \geq h(k) - 3\).

(ii) All the \(-(2r-i+1)\)-quasichords, \(-(2r-i+2)\)-quasichords and \(-(2r-i+3)\)-quasichords are in \(T\).

Proof: Assume that the hypothesis of the lemma holds and (i) is false. Let us prove that (ii) holds.

Since all the \([(k-3+i)]\)-quasichords and all the \([(k-2+i)]\)-quasichords are in \(T\), it follows from Lemma 6.1 (taking \(\alpha = i - 1\)) that every \([k-3+i+(t-1)(k-3)]\)-quasichord is in \(T\), and that (taking \(\alpha = i\)) every \([k-2+i+(t-1)(k-3)]\)-quasichord is in \(T\). Thus the following ordered pairs are quasichords of \(T\): \((r,0)\), \((r+1,0)\), \((0,k-2+(t-1)(k-3)+i)\), \((0,k-2+(t-1)(k-3)+i-1)\).

Let: \(x_1 = r, x_2 = r+1, x_3 = k-2+(t-1)(k-3)+i-1, x_4 = x_3+1, x_5 = x_4+k-3, x_6 = x_5+1, x_7 = x_5+1 \) and \(x_8 = x_7-1\). Thus, \((x_1,0)\) is a \((-r)\)-quasichord and \((0,x_4)\) is a \([(k-2)+(t-1)(k-3)+i]\)-quasichord. Observe that:

- It follows from \(x_5 = k-2+t(k-3)+i\), and \(n = k-2+t(k-3)+r\) that \(l(x_5,\gamma,0) = n-x_5=r-i\).
- \(l(x_6,\gamma,0) = r-i-1\).
- \(l(x_6,\gamma,x_1) = 2r-i-1\).
- \(l(x_7,\gamma,x_1) = 2r-i+1\).
- \(l(x_7,\gamma,x_2) = 2r-i+2\).
- \(l(x_8,\gamma,x_2) = 2r-i+3\).
- \(l(x_4,\gamma,x_1) = k-4\).
- \(l(x_3,\gamma,x_8) = k-4\).

I. We first prove that every \(-(2r-i+1)\)-quasichord is in \(T\). Suppose that not every \(-(2r-i+1)\)-quasichord is in \(T\). We can assume w.l.o.g. that \((x_7,x_1)\) is not a \(-(2r-i+1)\)-quasichord. Then by Remark 1 we have that \(\{(x_7,x_1),(x_5,x_1),(x_8,x_1)\} \cap A \neq \emptyset\). We consider the three possible cases.

Case 1: \((x_7,x_1) \in A\). In this case \(Q_{k-1} = (x_7,x_1,0,x_4) \cup (x_4,\gamma,x_7)\) is a 3-quasicycle with at most two pairs of consecutive vertices that are not necessarily arcs of \(T\), namely, \(f_1 = (x_1,0)\) and \(f_2 = (0,x_4)\). Notice that \(l(T_1) = 0\) and \(l(T_2) \neq 0\) \((T_1,T_2\) defined in Definition 2.4). It
follows from Lemma 2.3 that there exists a cycle $C_{h(l)}$, $h(l) \in \{l + 1, l + 2, l + 3, l + 4\}$, with $T(C_{h(l)}) \geq h(l) - 3$, and $l = k - 4$. Notice that the case $h(l) = l + 5$ in Lemma 2.3 does not occur because for this it is necessary that $\ell(T_1) = 0$ and $\ell(T_2) = 0$ (see Case 3 in the proof of the lemma).

**Case 2:** $(x_5, x_1) \in A$. Recall that $(x_1, 0)$ is a $(-r)$-quasichord. Hence we consider two possible subcases.

**Case 2.1:** $x_1$ is adjacent to 0. Then $(x_1, 0) \in A$ and $Q_k = (x_5, x_1, 0, x_4) \cup (x_4, \gamma, x_5)$ is a 3-quasicycle with at most one pair of consecutive vertices that is not necessarily and arc of $T$, namely, $f_3 = (0, x_4)$. Since $\ell(T_3) > 0$ it follows from Lemma 2.3 that there exists a cycle $C_{h(l)}$, $h(l) \in \{l + 1, l + 2\}$, with $T(C_{h(l)}) \geq h(l) - 3$, and $l = k - 3$.

**Case 2.2:** $x_1$ is not adjacent to 0. It follows from the definition of quasichord that $(x_1, 1)$ and $(x_1, n - 1)$ are in $A$. Also, $(x_4, x_1 + 1)$ is a $[(k - 2) + i + (t - 1)(k - 3)]$-quasichord. And then, $Q_{k-1} = (x_5, x_1, 1, x_4 + 1) \cup (x_4, \gamma, x_5)$ is a 3-quasicycle with at most one pair of consecutive vertices that is not necessarily and arc of $T$, namely, $f_1 = (1, x_4 + 1)$. Notice that $\ell(T_1) > 0$ and $k \geq 6$. It follows from Lemma 2.3 that there exists a cycle $C_{h(l)}$, $h(l) \in \{l + 3, l + 2\}$, with $T(C_{h(l)}) \geq h(l) - 3$, and $l = k - 4$.

**Case 3:** $(x_8, x_1) \in A$. Consider $Q_{k-2} = (x_8, x_1, 0, x_4) \cup (x_4, \gamma, x_8)$ is a 3-quasicycle with at most two pairs of consecutive vertices that are not necessarily arcs of $T$, namely, $f_1 = (x_1, 0)$, and $f_2 = (0, x_2)$. Notice that $\ell(T_1) = 0$ and $\ell(T_2) > 0$. It follows from Lemma 2.3 that there exists a cycle $C_{h(l)}$, $h(l) \in \{l + 2, l + 3, l + 4, l + 5\}$, with $T(C_{h(l)}) \geq h(l) - 3$, and $l = k - 5$.

**II.** Now we prove that every $-(2r-i+2)$-quasichord is in $T$. If not every $-(2r-i+2)$-quasichord is in $T$, we may assume without loss of generality that $(x_2, x_7)$ is not a $-(2r-i+2)$-quasichord. It follows from Remark 1 that at least one of the following holds: $(x_7, x_2) \in A$, $(x_8, x_2) \in A$, $(x_5, x_2) \in A$. We proceed as in the proof of I., changing $x_1$ for $x_2$.

**III.** Now we prove that every $-(2r-i+3)$-quasichord is in $T$. If not every $-(2r-i+3)$-quasichord is in $T$, we may assume without loss of generality that $(x_2, x_8)$ is not a $-(2r-i+3)$-quasichord. It follows from Remark 1 that at least one of the following holds: $(x_8, x_2) \in A$, $(x_7, x_2) \in A$, $(x_8 - 1, x_2) \in A$. We proceed as in the proof of I., changing $x_4$ for $x_3$.

**Lemma 6.4** At least one of the following properties holds.

(i) There exists a cycle $C_{h(k)}$, $h(k) \in \{k - 2, k - 1, k\}$ such that $T(C_{h(k)}) \geq h(k) - 3$.

(ii) For any $x$, there exist at most $k - 4$ consecutive vertices in $\gamma$, say $x_1, \ldots, x_m$, such that for all $1 \leq i \leq m$, $(x_i, x)$ is a quasichord of $\gamma$.

**Proof:** Assume that (i) does not hold, and assume for contradiction that (ii) does not hold. Let $x = 0$, without loss of generality. Thus there exist $k - 3$ consecutive vertices in $\gamma$, say
Lemma 6.5 If \( x_0, x_1, x_2 \) are vertices of \( T \) such that \( 0 < x_1 < x_0 < x_2 < n \), \( (x_1, x_2) \) is not a quasichord of \( \gamma \), \((x_1, 0)\) and \((0, x_0)\) are quasichords of \( \gamma \), and \( l(x_0, \gamma, x_2) = k - 5 \), then there exists a cycle \( C_{h(k)} \), \( h(k) \in \{k - 3, k - 2, k - 1, k\} \) such that \( I(C_{h(k)}) \geq h(k) - 3 \).

**Proof:** Since \((x_1, x_2)\) is not a quasichord of \( \gamma \), it follows from Remark 1 that at least one of the following holds: \((x_2, x_1) \in A, (x_2 - 1, x_1) \in A, (x_2 + 1, x_1) \in A\). We will consider the three possible cases.

**Case 1:** \((x_2, x_1) \in A\). In this case \( Q_{k-2} = (x_2, x_1, x_0, x_3) \cup (x_3, \gamma, x_2) \) is a 3-quasicycle, with one quasichord that is a chord of \( A \) (namely, \((x_2, x_1))\). It follows from Lemma 2.3 taking \( f_1 = (x_2, x_1), f_2 = (x_1, x_0), f_3 = (x_0, x_3) \), \( I(T_1) = l(T_2) = 0, l = k - 5 \), that there exists a cycle \( C_{h(k)} \), \( h(k) \in \{k - 3, k - 2, k - 1, k\} \) such that \( I(C_{h(k)}) \geq h(k) - 3 \) (notice that the first subcase of Case 3 in the proof of the lemma does not occur because \( I(T_1) = l(T_2) = 0 \), which is the only subcase which would give \( h(k) = k - 4 \)).

**Case 2:** \((x_2 + 1, x_1) \in A\). In this case \( Q_{k-1} = (x_2 + 1, x_1, x_0, x_3) \cup (x_3, \gamma, x_2 + 1) \) is a 3-quasicycle, with quasichords \( f_1 = (x_2 + 1, x_1), f_2 = (x_1, x_0), f_3 = (x_0, x_3) \). Notice that \( l(T_1) = l(T_2) = 0 \) and \( l(T_3) > 0 \) (since \( k > 5 \)). It follows from Lemma 2.3 taking the three quasichords \( f_1, f_2, f_3 \), \( l = k - 4 \), that there exists a cycle \( C_{h(k)} \), \( h(k) \in \{k - 3, k - 2, k - 1, k\} \) such that \( I(C_{h(k)}) \geq h(k) - 3 \) (notice that we don't obtain length \( l + 5 \) since \( f_1 \in A \) and \( l(T_3) > 0 \)).

**Case 3:** \((x_2 - 1, x_1) \in A\). In this case \( Q_{k-2} = (x_2 - 1, x_1, x_0, x_3) \cup (x_3, \gamma, x_2 - 1) \) is a 3-quasicycle, with quasichords \( f_1 = (x_2 - 1, x_1), f_2 = (x_1, x_0), f_3 = (x_0, x_3) \). It follows from Lemma 2.3 taking the three quasichords \( f_1, f_2, f_3 \), \( l = k - 6 \), that there exists a cycle \( C_{h(k)} \), \( h(k) \in \{k - 3, k - 2, k - 1\} \) except when: (a) We are in the second case of the proof of the lemma; when exactly one \( f_i \notin A \) and \( l(T_i) > 1 \). But the only possible such quasichord is \( f_3 \), and in this case we consider \( C_{k-2} \), the cycle of \( \gamma \) induced by the chords \( \{(x_2 - 1, x_1), (x_1, x_0), (x_0, x_3 - 1)\} \), and clearly \( I(C_{k-2}) = k - 5 \). (b) We are in the first subcase of the 3rd case of the proof of the Lemma; i.e., that there exist two quasichords \( g_1, g_2 \) such that \( g_i \notin A \) and \( l(T_i) > 0 \), \( i = 1, 2 \). Clearly this case is impossible.

Lemma 6.6 If every \( k \)-quasichord and every \((-r)\)-quasichord are in \( T \) then at least one of the two following properties holds.

(i) There exists a cycle \( C_{h(k)} \), \( h(k) \in \{k - 3, k - 2, k - 1, k\} \) such that \( I(C_{h(k)}) \geq h(k) - 3 \).
(ii) For every $\alpha$, $0 < \alpha(r+1) < k$, every $-(\alpha+1)(r+1)$-quasichord is in $T$.

**Proof:** Assume that (ii) does not hold; we show that (i) holds. Let $\alpha$ be the least integer for which there does not exist an $-(\alpha+1)(r+1)$-quasichord. By Lemma 6.2, $\alpha > 0$. It follows that there exist vertices $x_1, x_2$ such that $l(x_1, x_2, \gamma, x_1) = (\alpha+1)(r+1)$ and $(x_1, x_2)$ is not a quasichord of $\gamma$. Let $x_0 \in V$ such that $l(x_0, \gamma, x_0) = r+1$. It follows from the choice of $\alpha$ that $(x_1, x_0)$ is a quasichord (because it is an $-\alpha(r+1)$-quasichord). Let $x_3 \in V$ such that $l(x_0, \gamma, x_3) = k-1 + (t-1)(k-3)$. Observe that $x_3 \in \langle x_1, \gamma, x_0 \rangle$ because $\alpha(r+1) < k$ and $t \geq 1$.

Lemma 6.1 and the fact that every $(k-1)$-chord is in $A$ imply that either (i) holds or that every $k-1+(t-1)(k-3)$-quasichord is in $T$. Thus we can assume that every $k-1+(t-1)(k-3)$-quasichord is in $T$. Hence we have that $(x_0, x_3)$ is a quasichord of $\gamma$. The lemma follows from Lemma 6.5 replacing $x_1$ by $x_1$, $0$ by $x_0$, $x_0$ by $x_3$.

**Lemma 6.7** At least one of the following properties holds.

(i) There exists a cycle $C_{h(k)}$, $h(k) \in \{k-3, k-2, k-1, k\}$ such that $T(C_{h(k)}) \geq h(k) - 3$.

(ii) For $-1 \leq i \leq t$, every $-(2r+2-i)$-quasichord and every $(k-2+i)$-quasichord is in $A$.

**Proof:** Suppose that (i) does not hold. We shall prove that property (ii) holds by induction on $i$. We start with $i = -1$ and $i = 0$, namely, we prove that the following are quasichords of $\gamma$:

(a) every $(k-3)$-quasichord,

(b) every $(k-2)$-quasichord,

(c) every $-(2r+3)$-quasichord,

(d) every $-(2r+2)$-quasichord.

In fact we also prove that the following are quasichords of $\gamma$:

(e) every $-(2r+4)$-quasichord.

The proof of (a) follows from Lemma 6.2(d), while the proof of (b) follows from Lemma 6.2(a).

Let $0$ be any vertex of $T$. By Lemma 6.2(b) and (c) we have that $(r, 0)$ and $(r+1, 0)$ are quasichords. It follows from Corollary 2.5 that the following is a quasichord $(0, k-2+(t-1)(k-3))$. And by Lemma 6.2(d), and Lemma 2.4 that $(0, t(k-3))$ is a quasichord. Notice that these two quasichords have consecutive end-points in $\gamma$. 

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Since 0 is an arbitrary vertex of $T$, we can prove that (c), (d) and (e) hold:

**Part (c):** every $-(2r + 3)$-quasichord is in $T$. Assume $r + 1, n - r - 2$ is not a quasichord. Let $x_2 = n - (r + 2)$, $x_1 = r + 1$, and $x_0 = k - 2 + (t - 1)(k - 3)$. We have already noticed that $(0, x_0)$ and $(x_1, 0)$ are quasichords. Observe that $l(x_0, x_2) = k - 5$. Then the claim follows from Lemma 6.5.

**Part (d):** every $-(2r + 2)$-quasichord is in $T$. Assume $(r, n - (r + 2))$ is not a quasichord. Let $x_2 = n - (r + 2)$, $x_1 = r$, and $x_0 = k - 2 + (t - 1)(k - 3)$. Then the claim follows from Lemma 6.5.

**Part (e):** every $-(2r + 4)$-quasichord is in $T$. Assume $(r + 1, n - (r + 3))$ is not a quasichord. Let $x_2 = n - (r + 3)$, $x_1 = r + 1$, and $x_0 = t(k - 3)$. Then the claim follows from Lemma 6.5.

Assume that the lemma holds for each $i', i' \leq i$ and we prove it for $i + 1$; namely, we prove:

(a) Every $(k - 1 + i)$-quasichord is in $T$,

(b) Every $-(2r + 1 - i)$-quasichord is in $T$.

**Proof of (a)**

It follows from the inductive hypothesis that for each $j$, $0 \leq j \leq i$, every $(k - 2) + j$-quasichord and every $(k - 3) + j$-quasichord is in $T$. Hence, by Lemmas 6.2 and 6.3, every $-(2r - j + 1)$-quasichord, $-(2r - j + 2)$-quasichord and every $-(2r - j + 3)$-quasichord is in $T$. That is, for each $j$, $0 \leq j \leq i + 2$, every $-(2r - j + 3)$-quasichord is in $T$. These are $(i + 3)$-quasichords with initial vertices consecutive in $\gamma$ (taking a fixed vertex $x$ and the $-(2r - j + 3)$-quasichords that end in $x$, where $0 \leq j \leq i + 2$).

Assume for contradiction that $(0, x_3)$ is not a $(k - 1 + i)$-quasichord. Let $x_0 = n - (2r - i - 1)$. Hence letting $x_2 = 2$, we have that $(x_2, x_0)$ is a $-(2r - (i - 1))$-quasichord (taking $j = i + 2$ in the previous assertion).

Let us show that $x_0 \in \langle x_3 + 1, \gamma, n - 1 \rangle$:

$$l(x_0, \gamma, 0) = 2r - i - 1,$$

$$l(x_3, \gamma, x_0) = n - (k + i - 1 + 2r - i - 1)$$

$$= k - 2 + t(k - 3) + r - (k + i - 1 + 2r - i - 1)$$

$$\geq t(k - 3) + r - 2r \geq k - 3 - r.$$

Since we are assuming $r \leq k - 4$ then $l(x_3, \gamma, x_0) \geq 1$. Hence $l(x_0, \gamma, 0) \geq 1$, because $r \geq 1$.

Now, there exists an $x$ such that $x \in \langle x_2, \gamma, x_3 - 1 \rangle$ and such that $(x, x_0)$ is not a quasichord (this is a direct consequence of Lemma 6.4 and the fact that the number of vertices in $(x_2, \gamma, x_3 - 1)$ is at least $k - 3$). Let $x_4$ be the smallest (the nearest to 0 in $\gamma$) such vertex.
Let $x_1 = 0$. We will prove that $x_4 - i - 3 \in \langle x_1, \gamma, x_4 - 3 \rangle$. Since for each $j$, $0 \leq j \leq i + 2$, every $-(2x - j + 3)$-quasichord is in $T$, it follows that $\{(2, x_0), (3, x_0), \ldots, (i + 5, x_0)\}$ are quasichords. Hence, the election of $x_4$ implies $x_4 \geq i + 6$ and then $x_4 - i - 3 \geq 0 = x_1$.

Finally, since $l(x_4, \gamma, x_3) + l(x_1, \gamma, x_4 - i - 3) = k - 4$ then we consider the following $Q_{k-1} = \langle x_4 - i - 3, x_0, x_4 \rangle \cup \langle x_4, \gamma, x_3 \rangle \cup \langle x_3, x_1 \rangle \cup \langle x_1, \gamma, x_4 - i - 3 \rangle$. We prove the following

**Claim:** there exists a vertex $z \in \langle x_2, \gamma, x_3 - 1 \rangle$ such that $(x_0, z) \in A$.

Assume for contradiction that there is no such $z$. By the definition of $x_4$ we have that for each $y \in \langle 2, \gamma, x_4 - 1 \rangle$ it holds that if $y$ is adjacent to $x_0$ then $(y, x_0) \in A$. And by the assumption it holds that for each $y \in \langle x_4, \gamma, x_3 - 1 \rangle$ we have that if $y$ is adjacent to $x_0$ then $(y, x_0) \in A$. (Notice that $l(x_2, \gamma, x_3 - 1) \geq k - 3$, so we have $k - 2$ consecutive vertices $y \in \langle x_0 + 1, \gamma, x_3 \rangle$, such that if $y$ is adjacent to $x_0$ then $(y, x_0) \in A$.) Let $z$ be the first vertex in $\langle x_2, \gamma^{-1}, x_0 \rangle$ (where $\gamma^{-1}$ denotes the inverse traversal of $\gamma$) such that $(x_0, z) \in A$ (it exists since $(x_0, x_0 + 1) \in A$). Now consider $P = \langle x_0, z \rangle \cup \langle z, \gamma, z + k - 3 \rangle$; by the previous observation if $z + k - 3$ is adjacent to $x_0$ then $(z + k - 3, x_0) \in A$ and then $C_{k-1} = P \cup \langle z + k - 3, x_0 \rangle$ is a cycle of $T$ with $\mathcal{I}(C_{k-1}) \geq k - 4$. If $z + k - 3$ is not adjacent to $x_0$ then $z + k - 4$ is adjacent to $x_0$ and by the previous observation we have that $(z + k - 4, x_0) \in A$ $(k - 3 \geq 3$ since $k \geq 6)$ and then $C_{k-2} = \rangle x_0, z \rangle \cup \langle z, \gamma, z + k - 4 \rangle \cup \langle z + k - 4, x_0 \rangle$ is a cycle of $T$ with $\mathcal{I}(C_{k-2}) \geq k - 5$. Contradicting our initial assumption. Thus the previous claim holds.

Let $x_5$ be the first vertex in $\langle 2, \gamma, x_3 - 1 \rangle$ such that $(x_0, x_5) \in A$ (such a vertex exists by the previous Claim). Notice that for each $y \in \langle 2, \gamma, x_5 - 1 \rangle$ if $y$ is adjacent to $x_0$ then $(y, x_0) \in A$. Now consider $x_5 - i - 3$; since $l(x_1, \gamma, x_4 - i - 3) + l(x_4, \gamma, x_3) = k - 4$ we have that $l(x_1, \gamma, x_5 - i - 3) + l(x_5, \gamma, x_3) = k - 4$. Consider now the two possible cases:

**Case 1:** $x_5 - i - 3$ is adjacent to $x_0$. By the definition of $x_5$ we have that $(x_5 - i - 3, x_0) \in A$. Let $f_1 = (x_5 - 1, x_1)$, $f_2 = (x_3, x_1)$, and $f_3 = (x_5 + 1, x_1)$. Since $(x_1, x_3)$ is not a quasichord, we have that at least one $f_i \in A$, $i = 1, 2, 3$.

**Case 1.1:** $f_1 \in A$. Let $C_{k-2}$ the cycle of $T$ induced by $\{f_1, g_1, h\}$ where $g_1 = (x_5 - i - 3, x_0)$ and $h = (x_0, x_5)$. Clearly $\mathcal{I}(C_{k-2}) = k - 5$. A contradiction.

**Case 1.2:** $f_2 \in A$. Let $C_{k-1}$ the cycle of $T$ induced by $\{f_2, g_1, h\}$. Clearly $\mathcal{I}(C_{k-1}) = k - 4$. A contradiction.

**Case 1.3:** $f_3 \in A$. Let $C_k$ the cycle of $T$ induced by $\{f_3, g_1, h\}$. Clearly $\mathcal{I}(C_k) = k - 3$. A contradiction.

**Case 2:** $x_5 - i - 3$ is not adjacent to $x_0$. Clearly we have that $x_5 - i - 4$ is adjacent to $x_0$. By the definition of $x_5$ we have that $(x_5 - i - 4, x_0) \in A$. Let $g_2 = (x_5 - i - 4, x_0)$. Since $(x_1, x_3)$ is not a quasichord we have the following three possibilities:

**Case 2.1:** $f_1 = (x_3 - 1, 0) \in A$. Let $C_{k-3}$ the cycle of $T$ induced by $\{f_1, g_2, h\}$. Clearly $\mathcal{I}(C_{k-3}) = k - 6$. A contradiction.

**Case 2.2:** $f_2 = (x_3, x_1) \in A$. Let $C_{k-2}$ the cycle of $T$ induced by $\{f_2, g_2, h\}$. Clearly $\mathcal{I}(C_{k-2}) = k - 5$. A contradiction.
Case 2.3: $f_3 = (x_3 + 1, x_1) \in A$. Let $C_{k-1}$ the cycle of $T$ induced by $\{f_3, g_2, h\}$. Clearly $I(C_{k-1}) = k - 4$. A contradiction.

**Proof of (β)**

Part (β) follows from Lemma 6.3 (taking $i + 1$ instead of $i$) and the following facts.

- Every $(k - 2 + i')$-quasichord is in $T$ ($-1 \leq i' \leq i$). Follows from the induction hypothesis.
- Every $(k - 1 + i')$-quasichord is in $T$ ($-1 \leq i' \leq i$). Follows from part (α).
- Every $(-r)$-quasichord and every $-(r + 1)$-quasichord is in $T$. Follows from Lemma 6.2.

**Theorem 6.8** If $n \geq 2k - 4$ then there exists a cycle $C_{h(k)}$, $h(k) \in \{k - 3, k - 2, k - 1, k\}$ such that $I(C_{h(k)}) \geq h(k) - 3$.

**Proof:** The case of $n = 2k - 4$ is considered in Section 4. Assume that $n > 2k - 4$ and assume for contradiction that there is no such cycle.

It follows from Lemma 6.7 that for each $i$, $-1 \leq i \leq r$, every $-(2r + 2 - i)$-quasichord and every $(k - 2 + i)$-quasichord is in $T$. In particular all the following pairs are quasichords of $T$:

$$\{(0, k - 3), (0, k - 2), (0, k - 1), (0, k), \ldots, (0, k + r - 2)\}.$$ (1)

It follows from Lemma 6.2 that every $(-r)$-quasichord is in $T$, and by Lemma 6.7 that every $k$-quasichord is in $T$ (taking $i = 2$). Therefore, by Lemma 6.6 for every $\alpha$, $0 < \alpha(r + 1) < k$, every $-(\alpha + 1)(r + 1)$-quasichord is in $T$. Let $\alpha_0 = \max\{\alpha \in \mathcal{N} | \alpha(r + 1) < k\}$. Clearly $0 < \alpha_0(r + 1) < k$, and by Lemma 6.6 every $-(\alpha_0 + 1)(r + 1)$-quasichord is in $T$. In particular $((\alpha_0 + 1)(r + 1), 0)$ is a quasichord of $T$, and $k \leq (\alpha_0 + 1)(r + 1) < k + r$. Thus $y = (\alpha_0 + 1)(r + 1) \in \{k, k + 1, \ldots, k + r - 1\}$. Therefore $(y, 0)$ is a quasichord of $T$. On the other hand, (1) implies that $(0, y)$ is a quasichord of $T$. It follows that 0 and $y$ are not adjacent. Now, since $(0, y)$ is a quasichord of $T$, $(0, y + 1), (0, y - 1) \in A$, and since $(y, 0)$ is a quasichord of $T$, $(y, 1), (y, n - 1) \in A$. Also, since $(\alpha_0 + 1)(r + 1) \in \{k, k + 1, \ldots, k + r - 1\}$ we have that $(1, y)$ is a quasichord of $T$ (notice that $\{(1, k - 2), (1, k - 1), (1, k), \ldots, (1, k + r - 1)\}$ are quasichords of $T$). But 1 is adjacent to $y$, and $(y, 1) \in A$, a contradiction.

**Remark 2**

The bound of this theorem is tight, since it coincides with the upper bounds proved in [8] for tournaments (multipartite with one vertex per part) and in [6] for bipartite tournaments, where it is proved that in some cases there are no cycles with larger intersection with $\gamma$.  

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Also, in general it is not possible to prove the result of this theorem for a set of options smaller than four (those of the set \( \{k - 3, k - 2, k - 1, k\} \)). To see this, consider the cyclically 4-partite tournaments defined as follows. A tournament \( T \) has vertices \( V(T) = V_0 \cup V_1 \cup V_2 \cup V_3 \) and \((x, y)\) is an arc of \( T \) iff \( x \in V_i, y \in V_{i+1} \) (modulo 4), for \( i \in \{0, 1, 2, 3\} \). If \( k \equiv i \pmod{4} \) then all the cycles of \( T \) have length congruent with \( k - i \pmod{4} \).

References


