Independent sets and non-augmentable paths in generalizations of tournaments

H. Galeana-Sánchez and R. Gómez

Instituto de Matemáticas, U.N.A.M.
Circuito Exterior, Ciudad Universitaria C.P. 04510, México D.F., México

Abstract

We study different classes of digraphs, which are generalizations of tournaments, to have the property of possessing a maximal independent set intersecting every non-augmentable or every longest path. The classes are the arc-local tournament, quasi-transitive, path-mergeable, locally in- semicomplete (out-semicomplete), and semicomplete $k$-partite digraphs. A short survey is included in the introduction.

Keywords: Independent set; non-augmentable path; longest path; generalization of tournament

1 Introduction

The conjecture of Laborde, Payan and Xuong can be stated as follows: In every digraph, there exists a maximal independent set that intersects every longest path (see [19]). The conjecture is true for every digraph having a kernel, that is, an independent and absorbing set of vertices, e.g., every transitive digraph (many other classes of digraphs have kernels, for instance, see [10], [13]). In [19], Laborde, Payan and Xuong showed that in every symmetric digraph, there exists an independent set intersecting every longest path and with the property that each of its vertices is the origin of a longest path (they conjectured that this holds for all digraphs). In [8], Bang-Jensen, Huang and Prisner proved that every strongly connected (i.e. strong) locally in-semicomplete digraph has a hamiltonian cycle (hence every longest path intersects every independent set). They showed that a locally in-semicomplete digraph has a hamiltonian path if and only if it contains a vertex that can be reached by all other vertices by a directed path, a result that constituted a sufficient condition for any independent set to intersect every longest path for this class of digraphs. In [14], Galeana-Sánchez and Rincón-Mejía proved several sufficient conditions for a digraph to have an independent set intersecting every longest path. Later, in [12], Galeana-Sánchez investigates sufficient conditions for a digraph to have the property that each of its induced subdigraphs has a maximal independent set intersecting all its non-augmentable paths. Moreover, Galeana-Sánchez finds
necessary and sufficient conditions for this property to hold in case that the digraph is asymmetrical, and also finds necessary and sufficient conditions for any orientation of a graph to have this property. More recently, in [18], F. Havet proved that if a digraph has stability number at most two, then there exists a stable set that intersects every longest path. Here, the stability number is the cardinality of a largest stable set, i.e., the cardinality of a largest independent set.

In this paper, we show that the Laborde, Payan and Xuong conjecture is true for arc-local tournament digraphs, line digraphs, quasi-transitive digraphs, path-mergeable digraphs, in-semi-complete (out-semi-complete) digraphs, and semi-complete $k$-partite digraphs, all of them being generalizations of tournaments except for line digraphs (see [6]). We prove that there always exists a maximal independent set intersecting every non-augmentable path in a semi-complete digraph (condition 3.4). For arc-local tournament digraphs (section 2.1), we show that there exists a maximal independent set that intersects every non-augmentable path (theorem 2.14). Actually, we show that in an arc-local tournament digraph, every maximal independent set intersects every non-augmentable path of even length, and exhibit arc local tournament digraphs with maximal independent sets and non-augmentable paths of arbitrary odd length which do not intersect (proposition 2.16). We show that line digraphs satisfy a condition quite similar to the one defining arc local tournament digraphs (condition 2.18), and prove that every maximal independent set in a digraph satisfying this condition intersects every non-augmentable path (theorem 2.19). For quasi-transitive digraphs (section 3.3), using a structural theorem of Bang-Jensen and Huang taken from [7] (theorem 3.6 in this paper), we show that there exists a maximal independent set that intersects every non-augmentable path (theorem 3.9). Moreover, we show that if the quasi-transitive digraph is strong, then this maximal independent set has a natural decomposition according to Bang-Jensen and Huang’s structural theorem. For path-mergeable digraphs (section 4.2), we show that a longest path in a path-mergeable digraph is strongly internally and finally non-augmentable (definition 4.2). Then we show that in any strong digraph, every maximal independent set intersects every strongly internally and finally non-augmentable path (proposition 4.3). Next, we show that in any strong digraph, there exists a maximal independent set intersecting every strongly internally and finally non-augmentable path (theorem 4.5). Finally, we show that in any digraph, there exists a maximal independent set intersecting every strongly internally and finally non-augmentable path (theorem 4.6). For locally in-semi-complete (out-semi-complete) digraphs and semi-complete $k$-partition digraphs (sections 5.2 and 6.2), we show that both have the property of possessing a maximal independent set that intersects every non-augmentable path (theorems 5.3, 5.4 and 6.2).

We have taken all these classes of digraphs from a survey by Bang-Jensen and Gutin (see [6]). We refer the reader to it for a detailed exposition of results concerning them and restrict ourselves to briefly present the following summary. Arc-local tournament digraphs were introduced by Bang-Jensen in [1] as an extension of the idea of a generalization of semi-complete digraphs called locally semi-complete digraphs. Some properties of arc local tournament digraphs have
been studied by Bang-Jensen in [1] and [4], and by Bang-Jensen and Gutin in [6].

Galeana-Sánchez characterized all kernel-perfect and critical kernel-imperfect arc local tournament digraphs in [11], both classes introduced by Berge and Duchet in [9]. Quasi-transitive digraphs were introduced by Ghouila-Houri (see [15]). They are related to comparability digraphs in the sense that a graph can be oriented as a quasi-transitive digraph if and only if it is a comparability digraph. In [7], Bang-Jensen and Huang extensively study quasi-transitive digraphs. Path-mergeable digraphs were introduced by Bang-Jensen in [2]. They can be recognized in polynomial time and the merging of two internally disjoint paths can be done in a particular nice way in the sense that it is always possible to respect the order of one of the paths. Locally in-semicomplete (out-semicomplete) digraphs were introduced by Bang-Jensen in [3], and in [2] he proved that locally in-semicomplete (out-semicomplete) digraphs are path mergeable (in particular, every tournament is path-mergeable). Semicomplete k-partite digraphs have been recently studied. In [11], Gutin presents a survey on this kind of digraphs. See [5] for a unified and comprehensive survey on digraphs.

2 Arc local tournament digraphs

In this paper, a digraph $D$ will consist of a vertex set $V(D)$ and an arc set $A(D) \subset V(D) \times V(D)$. All digraphs will be simple, that is, there will be no loops nor multiple arcs between any pair of distinct vertices. For $u, v \in V(D)$, we will write $\overrightarrow{uv}$ or $\overrightarrow{vu}$ if $(u,v) \in A(D)$, and also, we will write $\overrightarrow{vu}$ if $\overrightarrow{uv}$ or $\overrightarrow{vu}$.

Definition 2.1. An independent set in a digraph $D$ is a subset of vertices $I \subset V(D)$ with no $x,y \in I$ such that $\overrightarrow{xy}$, and is maximal if there exists no $z \in V(D) - I$ such that $I \cup \{z\}$ is an independent set.

Definition 2.2. A path in a digraph $D$ is a finite sequence of distinct vertices $\gamma = (x_0, \ldots, x_n)$ such that $\overrightarrow{x_{i-1}x_i}$ for every $1 \leq i \leq n$, and its length is $n$ (zero-length path consists of a single vertex). We let $V(\gamma) = \{x_0, \ldots, x_n\}$.

Definition 2.3. A path $\gamma = (x_0, \ldots, x_n)$ in a digraph $D$ is non-augmentable if there exists no path $(y_0, \ldots, y_k)$ with $y_0, \ldots, y_k \in V(D) - V(\gamma)$ and such that $\overrightarrow{y_kx_0}$, $\overrightarrow{x_0y_0}$, $\overrightarrow{x_{i-1}y_0}$ and $\overrightarrow{y_kx_i}$ for some $1 \leq i \leq n$. More generally, $\gamma$ is non-augmentable if there exist no path $(z_0, \ldots, z_m)$ in $D$ with $m > n$, a function $\sigma : \{0, \ldots, n\} \to \{0, \ldots, m\}$ and $0 \leq r \leq n$ such that $x_i = z_{\sigma(i)}$ for every $0 \leq i \leq n$, $\sigma(i) \leq \sigma(i+1)$ for every $i \neq r$, and $\sigma(n) < \sigma(0)$ if $r < n$. Otherwise, $\gamma$ is augmentable.

Remark 2.4. The first definition of non-augmentability is a particular case of the second one, with $m = n + k + 1$, $r = n$ and $\sigma(i) = i + k + 1$ for every $0 \leq i \leq n$ if $y_kx_0$, $\sigma(i) = i$ for every $0 \leq i \leq n$ if $x_0y_0$, and $\sigma(j) = j$ for $j < i$ and $\sigma(j) = j + k + 1$ for $j \geq i$ if $x_{i-1}y_0$ and $y_kx_i$ for some $1 \leq i \leq n$, so that if $\gamma$ is augmentable, then $(y_0, \ldots, y_k, x_0, \ldots, x_n)$, or $(x_0, \ldots, x_n, y_0, \ldots, y_k)$, or $(x_0, \ldots, x_{i-1}, y_0, \ldots, y_k, x_i, \ldots, x_n)$ are paths in $D$. 
Definition 2.5. A path $\gamma$ in a digraph $D$ is a longest path if there exists no path in $D$ of bigger length.

Definition 2.6. A path $\gamma$ in a digraph $D$ and a subset of vertices $I \subset V(D)$ intersect if $V(\gamma) \cap I \neq \emptyset$, otherwise they do not intersect.

In this section, if $D$ is a digraph and $u, v, x, y \in V(D)$ are such that $uv$, $xu$ and $vy$, then we will write $xuvy$, and similarly, if $uv$, $xu$ and $vy$, then we will write $xyuv$.

Definition 2.7. A digraph $D$ is arc local tournament if whenever $u, v, x, y \in V(D)$ are such that $xuvy$ or $xyuv$, then $uv$.

Proposition 2.8. Let $D$ be an arc local tournament digraph. Let $I$ be a maximal independent set and let $\gamma = (x_0, \ldots, x_n)$ be a non-augmentable path in $D$ such that $V(\gamma) \cap I = \emptyset$. If there exists $z \in I$ such that $z\gamma_i$ for some $i = 0, \ldots, n$, then $i_0 = \min \{i \mid z\gamma_i\} = 1$.

Proof. If $i_0 \geq 2$, then $z\gamma_{i_0-2}$ because $z\gamma_{i_0-2}$ is minimal, $z\gamma_{i_0-2}$. There exists $y \in I$ such that $y\gamma_{i_0-1}$ because $I$ is a maximal independent set. Suppose that $y = z$. Then $\gamma_{i_0-1}$ because $i_0$ is minimal, and therefore $(x_0, \ldots, x_{i_0-1}, z, x_{i_0}, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that $y \neq z$. If $z\gamma_{i_0-1}$, then $\gamma z$ because $\gamma z$ is an independent set, and if $z\gamma_{i_0-1}$, then $\gamma z$ because $z\gamma_{i_0-2} \gamma_{i_0-1} y$, contradicting again that $I$ is an independent set. Now, if $i_0 = 0$, then $z\gamma_0$ and therefore $(z, x_0, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Hence $i_0 = 1$.

Definition 2.9. Let $D$ be a digraph. For every vertex $v \in V(D)$, let the in-degree of $v$ be the number of incoming edges to $v$ and let the out-degree of $v$ be the number of outgoing edges from $v$. Denote them by $\text{in}(v)$ and $\text{out}(v)$ respectively. We let $\mathcal{O}(D) = \{v \in V(D) \mid \text{out}(v) = 0\}$.

Lemma 2.10. Let $D$ be an arc local tournament digraph. Let $I$ be a maximal independent set and $\gamma = (x_0, \ldots, x_n)$ be a non-augmentable path in $D$ such that $V(\gamma) \cap I = \emptyset$. For every $z \in I$ such that $z\gamma_0$, we have $z \in I \cap \mathcal{O}(D)$.

Proof. Suppose that $\text{out}(z) > 0$. If $z\gamma_i$ for some $i = 0, \ldots, n$, then, by proposition 2.8, $z\gamma_1$ and hence $(x_0, z, x_1, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Let $w \in V(D) - V(\gamma)$ be such that $zw$. Then $\gamma_1$ because $w\gamma_0$. If $w\gamma_1$, then $(x_0, z, w, x_1, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that $w\gamma_1$. If $n = 1$, then $(x_0, x_1, w)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that $n > 1$. Then $\gamma_2$ because $\gamma_2$. In fact, $\gamma_2$ because otherwise, by proposition 2.8, $\gamma_1$ and hence $(x_0, z, x_1, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. If $n = 2$, then $(x_0, x_1, x_2, z)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that $n > 2$. Then $\gamma_3$ because $\gamma_3$ and hence $(x_0, x_1, x_2, z, w, x_3, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that $\gamma_3$. If $n = 3$, then $(x_0, x_1, x_2, x_3, w)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that $n > 3$. Then $\gamma_4$.
because $\overline{xy}x_4z_4$. In fact, $\overline{xy}x_4$ because otherwise, by proposition 2.8, $\overline{xy}x_4$ and hence $(x_0, z, x_1, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable.

If $n = 4$, then $(x_0, x_1, x_2, x_3, x_4, z)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Continue in this way. 

\end{proof}

\textbf{Corollary 2.11.} Let $D$ be an arc local tournament digraph. If $\mathcal{O}(D) = \emptyset$, then every maximal independent set intersects every non-augmentable path in $D$.

\textbf{Proposition 2.12.} Let $D$ be an arc local tournament digraph and let $D_0$ be the induced arc local tournament digraph that results from $D$ by removing the vertices in $\mathcal{O}(D)$. Let $I_0$ be a maximal independent set in $D_0$ and let $\gamma = (x_0, \ldots, x_n)$ be a non-augmentable path in $D$ such that $\{x_0, \ldots, x_r\} \cap I_0 = \emptyset$ for some $r < n$. If $z \in I_0$, then there exists no $i \leq r$ such that $\overline{z}x_i$.

\begin{proof}
First, observe that $x_i \in V(D_0)$ for every $0 \leq i < n$. Suppose that there exists $i \leq r$ such that $\overline{z}x_i$ and let $i_0 = \min\{i \mid \overline{z}x_i\}$. Suppose that $i_0 > 1$. Then $\overline{z}x_{i_0-2}$ because $x_{i_0-2}x_{i_0-1}x_{i_0}z$. In fact, $\overline{z}x_{i_0-2}$ because $i_0$ is minimal. Since $I_0$ is a maximal independent set in $D_0$, there exists $y \in I_0$ such that $\overline{yx}_{i_0-1}$. If $y = z$, then $\overline{z}x_{i_0-1}$ because $i_0$ is minimal, but then $(x_0, \ldots, x_{i_0-1}, z, x_{i_0}, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that $y \neq z$. If $\overline{yx}_{i_0-1}$, then $\overline{y}z$ because $\overline{z}x_{i_0-1}x_{i_0}z$, contradicting that $I_0$ is an independent set. If $\overline{yx}_{i_0-1}$, then $\overline{y}y$ because $\overline{z}x_{i_0-2}x_{i_0-1}y$, contradicting that $I_0$ is an independent set. If $i_0 = 0$, then $(z, x_0, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma_0$ is non-augmentable. Suppose that $i_0 = 1$. There exists $y \in I_0$ such that $\overline{yx}_0$ because $I_0$ is a maximal independent set in $D_0$. Suppose that $y = z$. In this case, $\overline{z}x_0$ because $i_0$ is minimal, but then $(x_0, z, x_1, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that $y \neq z$. If $\overline{yx}_0$, then $(y, x_0, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that $\overline{yx}_0$. Then $\overline{y}x_0$ because $\overline{z}x_{i_0-2}$ (observe that $n \geq 2$ because $n > r \geq i_0 = 1$). If $\overline{yx}_0$, then $\overline{y}x_0$ because $\overline{z}x_{i_0-2}y$, contradicting that $I_0$ is an independent set. Suppose that $\overline{y}x_0$. Since $y \in V(D_0)$, $out(y) > 0$ when we consider $y$ as a vertex of the digraph $D$. If there exists a vertex $w \in V(D) - V(\gamma)$ such that $\overline{y}w$, then $\overline{w}x_1$ because $w \overline{yx}_0x_1$, but then $\overline{y}w$ because $\overline{z}w_1x_1z$, contradicting that $I_0$ is an independent set. Hence there exists $s$ such that $0 \leq s \leq n$ and $\overline{y}x_s$. If $s = 0$, then $(y, x_0, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. If $s = 1$, then $(x_0, y, x_1, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. If $s > 1$, then $\overline{y}x_s$ because $\overline{y}x_1x_0x_s$, but then $\overline{y}x_s$ because $\overline{z}x_1x_s$, contradicting that $I_0$ is an independent set.

\end{proof}

\textbf{Lemma 2.13.} Let $D$ be an arc local tournament digraph and let $D_0$ be the induced arc local tournament digraph that results from $D$ by removing the vertices in $\mathcal{O}(D)$. If $I_0$ is a maximal independent set in $D_0$ that intersects every non-augmentable path in $D_0$, then $I_0$ intersects every non-augmentable path in $D$.

\begin{proof}
Let $\gamma$ be a non-augmentable path in $D$. If $V(\gamma) \cap \mathcal{O}(D) = \emptyset$, then $\gamma$ is a non-augmentable path in $D_0$ and hence $V(\gamma) \cap I_0 \neq \emptyset$. Henceforth we
assume that $\gamma$ ends in a vertex in $O(D)$. If $\gamma$ becomes a non-augmentable path in $D_0$ after removal of $O(D)$, then $V(\gamma) \cap I_0 \neq \emptyset$. Suppose that $\gamma$ does not become a non-augmentable path in $D_0$. Then there exists a non-augmentable path $\gamma_0 = (x_0, \ldots, x_n)$ in $D$ and $k \geq 0$ with $k < n$ such that $\gamma$ becomes the path $(x_0, \ldots, x_k)$ after removal of $O(D)$ (so the length of $\gamma$ is $k+1$). Since $\gamma(0) \cap I_0 \neq \emptyset$, there exists $i \geq 0$, with $i \leq n$, such that $x_i \in I_0$. Let $i_0 = \min\{i \mid x_i \in I_0\}$. If $i_0 \leq k$ (in particular if $i_0 = 0$), then $V(\gamma) \cap I_0 \neq \emptyset$. Henceforth we suppose that $i_0 > k$.

Let $r = i_0 - 1 < n$. Then $x_i \notin I_0$ for all $i = 0, \ldots, r$ and hence, by proposition 2.12, there exist no $i \leq r$ such that $\overline{zx_i}$ for every $z \in I_0$. Let $x \in O(D)$ be such that $\gamma = (x_0, \ldots, x_k, x)$. Suppose that $k \geq 1$. There exists $z \in I_0$ such that $\overline{zx_k-1}$ because $I_0$ is a maximal independent set in $D_0$. By proposition 2.12, $\overline{zx_k-1}$ since $k \leq r$. Then $\overline{zx}$ because $\overline{zx_k-1x_k}$. Since $x \in O(D)$, $\overline{zx}$. There exists $y \in I_0$ such that $\overline{yx_k}$ because $I_0$ is a maximal independent set in $D_0$. Suppose that $y = z$. By proposition 2.12, $\overline{zx_k}$ since $k \leq r$, but then $(x_0, \ldots, x_k, x)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that $y \neq z$. By proposition 2.12, $\overline{yx_k}$ because $k < r$, and hence $\overline{yx}$ because $\overline{zx_k-1x_k}$, contradicting that $I_0$ is an independent set. Suppose that $k = 0$ (hence $\gamma = (x_0, x)$). If $n \geq 2$, then $\overline{x_2}$ because $\overline{x_0x_1x_2x_3}$. Since $x \in O(D)$, $\overline{x_2}$, but then $(x_0, x_1, x_2, x)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that $n = 1$. Then $i_0 = 1$ and hence $x_1 \in I_0$. Now, $\text{out}(x_1) > 0$ when we consider $x_1$ as a vertex of the digraph $D$. If $\overline{x_1x_0}$, then $(x_1, x_0, x)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable (here we are using the general definition of non-augmentability in definition 2.3). If $\overline{x_3x}$, then $(x_0, x_1, x)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that there exists $w \in V(D) - V(\gamma)$ such that $\overline{x_1w}$. Then $\overline{xw}$ because $\overline{x_0x_1w}$, but since $x \in O(D)$, $\overline{xw}$. Then $(x_0, x_1, w, x)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable.

It follows that supposing $i_0 > k$ leads a contradiction. Hence $i_0 \leq k$ and therefore $V(\gamma) \cap I_0 \neq \emptyset$. \hfill \qed

**Theorem 2.14.** If $D$ is an arc local tournament digraph, then there exists a maximal independent set that intersects every non-augmentable path.

**Proof.** We proceed by induction on the number of vertices. Clearly, the theorem is true if $|V(D)| = 1$. Let $m > 1$ and suppose that the theorem is true for every arc local tournament digraph with $k < m$ vertices. Suppose that $|V(D)| = m$. Let $D_0$ be the induced arc local tournament digraph obtained from $D$ by removing the vertices in $O(D)$. By lemma 2.10, if $O(D) = \emptyset$, then every maximal independent set intersects every non-augmentable path in $D$. Suppose that $O(D) \neq \emptyset$. Then $|V(D_0)| < m$ and the induction hypothesis implies that there exists a maximal independent set $I_0 \subseteq V(D_0)$ in the digraph $D_0$ that intersects every non-augmentable path in $D_0$. By lemma 2.13, if $\gamma$ is a non-augmentable path in $D$, then $V(\gamma) \cap I_0 \neq \emptyset$. Hence any maximal independent set $I \subseteq V(D)$ containing $I_0$ intersects every non-augmentable path in $D$. \hfill \qed
Remark 2.15. Clearly, \( O(D) \) is an independent set. If every non-augmentable path in \( D \) ends in a vertex in \( O(D) \), then any maximal independent set \( I \subset V(D) \) containing \( O(D) \) intersects every non-augmentable path in \( D \). On the other hand, if no non-augmentable path in \( D \) ends in a vertex in \( O(D) \), then the set of non-augmentable paths in \( D_0 \) corresponds to the set of non-augmentable paths in \( D \), and hence any maximal independent set \( I \subset V(D) \) containing a maximal independent set \( I_0 \) intersecting every non-augmentable path in \( D_0 \) intersects every non-augmentable path in \( D \).

Proposition 2.16. The following statements are true.

1. In any arc local tournament digraph, every maximal independent set intersects every non-augmentable path of even length.

2. For every odd number \( n \), there exists an arc local tournament digraph in which there exist a maximal independent set and a non-augmentable path of length \( n \) which do not intersect.

Proof. Let \( D \) be an arc local tournament digraph. Let \( \gamma = (x_0, \ldots, x_n) \) be a non-augmentable path in \( D \) and let \( I \) be a maximal independent set. Suppose that \( V(\gamma) \cap I = \emptyset \). There exists \( y \in I \) such that \( \overrightarrow{yx_0} \) because \( I \) is a maximal independent set. If \( \overrightarrow{yx_0} \) then \( (y, x_0, \ldots, x_n) \) is a path in \( D \) contradicting that \( \gamma \) is non-augmentable, so \( \overrightarrow{yx_0} \). There exists \( z \in I \) such that \( \overrightarrow{zx_n} \) because \( I \) is a maximal independent set. If \( \overrightarrow{zx_n} \) then \( (x_0, x_n, \ldots, x_l, z) \) is a path in \( D \) contradicting that \( \gamma \) is non-augmentable, so \( \overrightarrow{zx_0} \). Suppose that \( n = 1 \). If \( y = z \), then \((x_0, y, x_1)\) is a path in \( D \) contradicting that \( \gamma \) is non-augmentable. If \( y \neq z \), then we obtain an arc local tournament digraph in which there is a maximal independent set \( I = \{y, z\} \) and a path \( \gamma = (x_0, x_1) \) of length 1 which do not intersect, so (2) is true for \( n = 1 \) (see figure 1).

![Figure 1: A digraph satisfying 2 in proposition 2.16 for n = 1.](image-url)

Suppose that \( n \geq 2 \). Then \( \overrightarrow{yx_2} \) because \( \overrightarrow{yx_0x_1x_2} \). In fact, \( \overrightarrow{yx_k} \) as long as \( \overrightarrow{yx_{k-2}} \) with \( k \geq 2 \) and \( k \leq n \) since \( \overrightarrow{yx_{k-2}x_{k-1}x_k} \). Suppose that \( \overrightarrow{yx_{k-2}} \) but \( \overrightarrow{yx_k} \). There exists \( w \in I \) such that \( \overrightarrow{wx_{k-1}} \) because \( I \) is a maximal independent set. Suppose that \( w = y \). If \( \overrightarrow{yx_{k-1}} \), then \((x_0, \ldots, x_{k-2}, y, x_{k-1}, \ldots, x_n)\) is a path in \( D \) contradicting that \( \gamma \) is non-augmentable, and if \( \overrightarrow{yx_{k-1}} \) then \((x_0, \ldots, x_{k-1}, y, x_{k}, \ldots, x_n)\) is a path in \( D \) contradicting that \( \gamma \) is non-augmentable. Suppose that \( w \neq y \). If \( \overrightarrow{wx_{k-1}} \), then \( \overrightarrow{yw} \) because \( \overrightarrow{wx_{k-1}x_ky} \), contradicting that \( I \) is an independent set, and if \( \overrightarrow{wx_{k-1}} \), then \( \overrightarrow{yw} \) because \( \overrightarrow{yx_{k-2}x_{k-1}w} \), contradicting that \( I \) is an independent set. For every \( k \leq n \) even, \( \overrightarrow{yx_k} \) because \( \overrightarrow{yx_0} \).
If $n$ is even, then $\overline{yx}$, and hence $(x_0, \ldots, x_n, y)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. So (1) follows. Figures 2 and 3 describe arc local tournament digraphs satisfying (2) for $n = 3$ and $n = 5$, and figure 4 describes the general construction for $n$ odd, with $I = \{y, z\}$ as the maximal independent set and $\gamma = (x_0, \ldots, x_n)$ as the non-augmentable path.

![Figure 2](image1.png)

Figure 2: A digraph satisfying 2 in proposition 2.16 for $n = 3$.

![Figure 3](image2.png)

Figure 3: A digraph satisfying 2 in proposition 2.16 for $n = 5$.

![Figure 4](image3.png)

Figure 4: A digraph satisfying 2 in proposition 2.16 for $n$ odd.

**Definition 2.17.** Let $D$ be a digraph. Let the line digraph of $D$ be the digraph $\mathcal{L}(D)$ with vertex set $V(\mathcal{L}(D)) = A(D)$ and arc set $A(\mathcal{L}(D))$ defined by the following rule. If $x, y, z \in V(D)$ are such that $\overline{xy}$ and $\overline{yz}$, then $(x, y)(y, z)$.

Line digraphs are similar to arc local tournament digraphs in the sense that line digraphs satisfy the following condition.
Condition 2.18. For a digraph $D$, whenever $u, v, x, y \in V(D)$ are such that $\overrightarrow{xu}$, $\overrightarrow{uv}$ and $\overrightarrow{vy}$, then $\overrightarrow{xy}$.

For $u, v, w, x, y \in V(D)$, if $(x, u)(u, v)$, $(u, v)(w, u)$ and $(w, u)(u, y)$, then an arrangement described in figure 5 occurs and therefore $(x, u)(u, y)$.

![Figure 5: Arrangement that results in the configuration described in condition 2.18 in a line digraph.]

Theorem 2.19. In any digraph satisfying condition 2.18, every maximal independent set intersects every non-augmentable path.

Proof. Let $D$ be a digraph satisfying condition 2.18. Suppose that $\mathcal{I}$ is a maximal independent set and $\gamma = (x_0, \ldots, x_n)$ is a non-augmentable path in $D$ such that $V(\gamma) \cap \mathcal{I} = \emptyset$. There exists $z \in \mathcal{I}$ such that $\overrightarrow{zx_0}$ because $\mathcal{I}$ is a maximal independent set. If $\overrightarrow{zx_0}$, then $(z, x_0, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that $\overrightarrow{zx_0}$. Let $k_0 = \max \{ k \mid \overrightarrow{zx_k} \}$. If $k_0 = n$, then $(x_0, \ldots, x_n, z)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that $k_0 < n$. There exists $y \in \mathcal{I}$ such that $\overrightarrow{yx_{k_0+1}}$ because $\mathcal{I}$ is a maximal independent set. If $y = z$, then $\overrightarrow{zx_{k_0+1}}$ because $k$ is maximal, and hence $(x_0, \ldots, x_{k_0}, z, x_{k_0+1}, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that $y \neq z$. If $\overrightarrow{yx_{k_0+1}}$, then $\overrightarrow{zx_k}$ because $\overrightarrow{zx_{k_0+1}}$ and $\overrightarrow{zx_k}$, contradicting that $\mathcal{I}$ is an independent set. Suppose that $\overrightarrow{yx_{k_0+1}}$ and repeat the argument as many times as necessary until we find an element $w \in \mathcal{I}$ such that $\overrightarrow{wx_{n-1}}$. Now, since $\mathcal{I}$ is a maximal independent set, there exists $u \in \mathcal{I}$, such that $\overrightarrow{uw_{n-1}}$. If $\overrightarrow{uw_{n-1}}$, then $(x_0, \ldots, x_n, u)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Suppose that $\overrightarrow{uw_{n-1}}$. If $u = w$, then $(x_0, \ldots, x_{n-1}, w, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. If $u \neq w$, then $\overrightarrow{w}w$ because $\overrightarrow{uw_{n-1}}, \overrightarrow{wx_{n-1}}$ and $\overrightarrow{x_{n-1}w}$, contradicting that $\mathcal{I}$ is an independent set.

Corollary 2.20. Let $D$ be a digraph. Then every maximal independent set $\mathcal{I} 
subset V(L(D))$ intersects every non-augmentable path in $L(D)$.

3 Quasi-transitive digraphs

Definition 3.1. A digraph $D$ is transitive if whenever $u, v, w \in V(D)$ are such that $\overrightarrow{uw}$ and $\overrightarrow{vw}$, then $\overrightarrow{uw}$. The digraph $D$ is quasi-transitive if whenever $u, v, w \in V(D)$ are such that $\overrightarrow{uw}$ and $\overrightarrow{vw}$, then $\overrightarrow{uw}$.
Definitions 3.2. Let $D$ be a digraph. If for every $u, v \in V(D)$ there exist a path that starts in $u$ and ends in $v$ and a path that starts in $v$ and ends in $u$, then $D$ is strong, otherwise is non-strong. The digraph $D$ is oriented if it contains no cycles of length two, that is, if there exist no $u, v \in V(D)$ such that $\overrightarrow{uv}$ and $\overrightarrow{vu}$. The digraph $D$ is semicomplete if $\overrightarrow{uv}$ for every $u, v \in V(D)$.

Lemma 3.3. If $D$ is a digraph and $\gamma$ is a non-augmentable path in $D$, then there exists no $z \in V(D) - V(\gamma)$ such that $\overrightarrow{xz}$ for every $x \in V(\gamma)$.

Proof. Suppose that $\gamma = (x_0, \ldots, x_n)$ and $z \in V(D) - V(\gamma)$ are such that $\overrightarrow{xz}$ for every $x \in V(\gamma)$. If $\overrightarrow{x_0z}$, then $(z, x_0, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable, so $\overleftrightarrow{x_0z}$. If $\overrightarrow{x_1z}$, then $(x_0, z, x_1, \ldots, x_n)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable, so $\overleftrightarrow{x_1z}$. Continuing in this way, it follows that $\overleftrightarrow{x_nz}$, but then $(x_0, \ldots, x_n, z)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. □

Proposition 3.4. Let $D$ be a semicomplete digraph. Then every maximal independent set consists of a single vertex and intersects every non-augmentable path.

Proof. Clearly, if $I \subseteq V(D)$ is a maximal independent set, then $|I| = 1$ because $D$ is semicomplete, say $I = \{z\}$ with $z \in V(D)$ arbitrary. Since $D$ is semicomplete, $\overrightarrow{xz}$ for every $x \in V(D) - \{z\}$. By lemma 3.3, $I$ intersects every non-augmentable path in $D$. □

Definition 3.5. Let $D$ be a digraph and let $\{\alpha_u\}_{u \in V(D)}$ be a family of digraphs indexed by $u \in V(D)$. The sum of $D$ and $\{\alpha_u\}_{u \in V(D)}$ is the digraph $\sigma(D, \alpha_u)$ with vertex set $\bigcup_{u \in V(D)} \{u\} \times V(\alpha_u)$, and for every $(u, x), (v, y) \in V(\sigma(D, \alpha_u))$, $(u, x)(v, y)$ if $u = v$ and $\overrightarrow{xv}$, or if $u \neq v$ and $\overrightarrow{uv}$.

Theorem 3.6 (Bang-Jensen and Huang [7]). Let $Q$ be a quasi-transitive digraph. There exist a digraph $D$ and a family of digraphs $\{\alpha_u\}_{u \in V(D)}$ such that $Q = \sigma(D, \alpha_u)$ and satisfying the following.

1. If $Q$ is non-strong, then $D$ is transitive oriented and $\alpha_u$ is strong quasi-transitive for all $u \in V(D)$.

2. If $Q$ is strong, then $D$ is strong semicomplete and $\alpha_u$ is non-strong quasi-transitive for all $u \in V(D)$.

For a quasi-transitive digraph $Q$, we will always write $Q = \sigma(D, \alpha_u)$ where $D$ and $\{\alpha_u\}_{u \in V(D)}$ are as in theorem 3.6.

Proposition 3.7. Let $H = \sigma(D, \alpha_u)$. If $D$ is transitive oriented and if for every $u \in V(D)$, there exists a maximal independent set $I_u \subset V(\alpha_u)$ that intersects every non-augmentable path in $\alpha_u$, then there exists a maximal independent set that intersects every non-augmentable path in $H$.  

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Proof. The digraph $D$ has a kernel because it is transitive. Therefore, there exists a maximal independent set $\mathcal{I} \subset V(D)$ that intersects every non-augmentable path in $D$. Clearly, $\mathcal{J} = \bigcup_{u \in \mathcal{I}} \{u\} \times \mathcal{I}_u$ is a maximal independent set. Let $\gamma = ((u_0, x^{(0)}_0), \ldots, (u_{m(0)}, x^{(m(0))}_0), \ldots, (u_n, x^{(n)}_n))$ be a non-augmentable path in $\sigma(D, \alpha_u)$. Then $u_{i-1}u_i$ for every $1 \leq i \leq n$, and hence $u_iu_j$ for all $i < j$ because $D$ is transitive. It follows that $u_i \neq u_j$ for all $i \neq j$ since $D$ is oriented. Therefore $\gamma_D = (u_0, \ldots, u_n)$ is a non-augmentable path in $D$ and $\gamma_i = (x_0^{(i)}, \ldots, x_n^{(i)})$ is a non-augmentable path in $\alpha_u_i$. Hence there exist $u = u_i \in V(\gamma_D) \cap \mathcal{I}$ and $x \in V(\gamma_i) \cap \mathcal{I}_u$, and thus $(u, x) \in V(\gamma) \cap \mathcal{J} \neq \emptyset$. □

Definition 3.8. Let $D$ be a digraph. A path $\gamma = (x_0, \ldots, x_n)$ in $D$ is internally non-augmentable with respect to $B \subset V(D) - V(\gamma)$ if there exists no path $(y_0, \ldots, y_r)$ in $D$ with $r \geq 0$ and $y_j \in B$ for every $0 \leq j \leq r$ and such that $\overline{x_{i-1}y_0}$ and $\overline{y_kx_i}$ for some $1 \leq i \leq n$.

Theorem 3.9. Let $Q = \sigma(D, \alpha_u)$ be a quasi-transitive digraph. Then there exists a maximal independent set $\mathcal{J} \subset V(Q)$ that intersects every non-augmentable path in $Q$. Moreover, if $Q$ is strong and $\mathcal{I}_u \subset V(\alpha_u)$ is a maximal independent set intersecting every non-augmentable path in $\alpha_u$, then $\mathcal{J} = \{u\} \times \mathcal{I}_u$ is a maximal independent set intersecting every non-augmentable path in $Q$.

Proof. We proceed by induction on the number of vertices. Clearly, the result is true if $|V(Q)| = 1, 2$. Suppose that the result is true for every quasi-transitive digraph with at most $m - 1$ vertices. Suppose that $|V(Q)| = m$.

If $Q$ is non-strong, then, by (1) in theorem 3.6, $D$ is transitive oriented and $\alpha_u$ is strong quasi-transitive with $|V(\alpha_u)| < m$ for every $u \in V(D)$. By the induction hypothesis, there exists a maximal independent set $\mathcal{I}_u \subset V(\alpha_u)$ that intersect every non-augmentable path in $\alpha_u$ for every $u \in V(D)$. Hence, in this case, the result follows from proposition 3.7.

If $Q$ is strong, then, by (2) in theorem 3.6, $D$ is strong semicomplete and $\alpha_u$ is non-strong quasi-transitive with $|V(\alpha_u)| < m$ for every $u \in V(D)$. By proposition 3.4, a maximal independent set is of the form $\mathcal{J} = \{u\} \times \mathcal{I}_u$ for some $u \in V(D)$ and some maximal independent set $\mathcal{I}_u \subset V(\alpha_u)$. By the induction hypothesis, we can suppose that $\mathcal{I}_u$ intersects every non-augmentable path in $\alpha_u$. Let $\gamma$ be a non-augmentable path in $Q$. If $V(\gamma) \cap (\{v\} \times V(\alpha_v)) = \emptyset$ for some $v \in V(D) - \{u\}$, then $(v, y)(w, z)$ for every $(w, z) \in V(\gamma)$ and $y \in V(\alpha_v)$, contradicting, by lemma 3.3, that $\gamma$ is non-augmentable. Then $\{u \in V(D) \mid (u, x) \in V(\gamma) \text{ for some } x \in V(\alpha_u)\} = V(D)$. Moreover, let $v \in V(D) - \{u\}$ and $y \in V(\alpha_v)$. Clearly, $(v, y) \in V(\gamma)$ if $|V(\alpha_v)| = 1$. Suppose that $|V(\alpha_v)| > 1$. Let $Q'$ be the strong quasi-transitive digraph that results from $Q$ by removing vertex $(v, y)$, so that $|V(Q')| < |V(Q)|$. If $(v, y) \notin V(\gamma)$, then $\gamma$ and $\mathcal{J}$ remain the same in $Q'$, and therefore, by the induction hypothesis, $V(\gamma) \cap \mathcal{J} \neq \emptyset$. So $\{y \in V(\alpha_v) \mid (v, y) \in V(\gamma)\} = V(\alpha_v)$ when $V(\gamma) \cap \mathcal{J} = \emptyset$. We will use the following lemma.

Lemma 3.10. Let $x, y \in V(\alpha_v) - \mathcal{I}_u$. If there exists a path $\rho$ of length at least two, starting at $(u, x) \in V(Q)$, ending at $(u, y) \in V(Q)$, with $\{z \in \mathcal{J} \mid (u, z) \in V(\gamma)\} \neq \emptyset$. □
V(\alpha_u) \mid (u, z) \in V(\rho) \text{ for some } z \in V(\alpha_u) = \{x, y\}, \text{ and internally non-augmentable with respect to } \{u\} \times I_u, \text{ then } \alpha_u \text{ is semicomplete.}

**Proof of lemma 3.10.** We proceed by induction on the length of \(\rho\). First, let \(\rho = ((u, x), (v_1, z_1), (u, y))\) be a path of length two in \(Q\), with \(v_1 \in V(D) - \{u\}\) and \(z_1 \in V(\alpha_{v_1})\). For every \(x', y' \in \alpha_u\), the definition of the sum implies that \(x' z_1\) and \(z_1 y'\), and then \(x'y'\) because \(Q\) is quasi-transitive. Suppose that the result is true for every path of length \(k \geq 2\) satisfying the hypothesis of the lemma. Let \(x = (u, x), y = (u, y)\) and \(z_i = (v_i, z_i)\) with \(v_i \in V(D) - \{u\}\) and \(z_i \in V(\alpha_{v_i})\) for every \(1 \leq i \leq k\) so that \(\rho = (x, z_1, \ldots, z_k, y)\) is a path of length \(k + 1\) satisfying the hypothesis of the lemma. Since \(\overline{z_{k-1}z_k}\) and \(\overline{z_ky}\), \(\overline{z_k-1y}\) because \(Q\) is quasi-transitive. If \(\overline{z_{k-1}y}\), then \(\rho' = (x, z_1, \ldots, z_{k-1}, y)\) is a path that starts at \(x = (u, x)\), ends at \(y = (u, y)\), with \(\{z \in V(\alpha_u) \mid (u, z) \in V(\rho') \text{ for some } z \in V(\alpha_u)\} = \{x, y\}\). Since \(I_u\) is independent, a path in \(D\) with its vertices in \(\{u\} \times I_u\) is necessarily a zero-length path. Suppose that \(y_0 \in \{u\} \times I_u\) is such that \(\overline{z_i-1y_0}\) and \(\overline{y_0z_i}\) for some \(1 < i < k\), or \(\overline{xy_0}\) and \(\overline{y_0z_1}\), or \(\overline{z_{k-1}y_0}\) and \(\overline{y_y}\). Clearly, having the first or the second of these cases holding contradicts that \(\rho\) is internally non-augmentable with respect to \(\{u\} \times I_u\). Suppose that \(\overline{z_{k-1}y_0}\) and \(\overline{y_y}\). Since \(\overline{z_ky}\) and \(v_k \neq u\), the definition of the sum implies that \(\overline{z_ky_0}\), contradicting, together with \(\overline{y_y}\), that \(\rho\) is internally non-augmentable with respect to \(\{u\} \times I_u\). Therefore \(\rho'\) is a path of length \(k \geq 2\) internally non-augmentable with respect to \(\{u\} \times I_u\) so that the induction hypothesis implies that \(\alpha_u\) is semicomplete. Suppose that \(\overline{z_{k-1}y}\). Suppose that \(\overline{y_0}\) for some \(j < k - 1\) and let \(j_0 = \max\{j \mid \overline{z_0y}\}\) so that \(\overline{z_{j_0+1}y}\). The definition of the sum implies that for every \(w \in I_u, \overline{z_{j_0}w}\) and \(\overline{z_{j_0+1}w}\), where \(w = (u, w)\), contradicting that \(\rho\) is internally non-augmentable with respect to \(\{u\} \times I_u\). Therefore \(\overline{z_{j_2}y}\) for every \(j < k\). In particular, \(\overline{z_{j_2}y}\) and hence the definition of the sum implies that \(\overline{z_{j_2}y}\). Then \(\rho'' = (x, z_2, \ldots, z_k, y)\) is a path of length \(k\) that starts at \(x = (u, x)\), ends at \(y = (u, y)\), with \(\{z \in V(\alpha_u) \mid (u, z) \in V(\rho') \text{ for some } z \in V(\alpha_u)\} = \{x, y\}\) and internally non-augmentable with respect to \(\{u\} \times I_u\) as shown by an argument similar to the one above, so that the induction hypothesis implies that \(\alpha_u\) is semicomplete.

Write \(\gamma = ((u_0, x_0^{(0)}), \ldots, (u_0, x_0^{(0)}), \ldots, (u_n, x_n^{(n)}), \ldots, (u_n, x_n^{(n)}))\). Suppose that \(u_i = u_j = u\) for some \(0 \leq i < j + 1 \leq n\) and \(u_k \neq u\) for every \(i < k < j\). The path obtained from \(\gamma\) that starts in \(x = (u_i, x_i^{(i)}) \in V(\alpha_u)\) and ends in \(y = (u_j, x_j^{(j)}) \in V(\alpha_u)\) satisfies the hypothesis of lemma 3.10, implying that \(\alpha_u\) is semicomplete. Therefore, by proposition 3.4, \(I_u = \{z\}\) for some \(z \in V(\alpha_u)\). Since \(I_u\) is a maximal independent set in \(\alpha_u\) and \(D\) is semicomplete, \((u, z)(u_i, x_i^{(i)})\) for every \(0 \leq i \leq n\) and \(0 \leq j \leq n(i)\), contradicting, by lemma 3.3, that \(\gamma\) is non-augmentable. It follows that there exists a unique \(i \in \{0, \ldots, n\}\) such that \(u_i = u\), and hence \((x_i^{(i)}), \ldots, x_i^{(i)}\) is a non-augmentable path in \(\alpha_u\) because otherwise \(\gamma\) would be augmentable. Therefore there exists \(0 \leq j \leq n(i)\) such that \(x_j^{(i)} \in I_0\) and hence \((u_i, x_j^{(i)}) \in J\), i.e., \(\gamma \cap J \neq \emptyset\). \(\square\)
4 Path-mergeable digraphs

Definition 4.1. A digraph $D$ is $k$-path-mergeable for some integer $k \geq 2$ if for every $u, v \in V(D)$ and any pair of paths $\gamma$ and $\rho$ of length at most $k$, starting at $u$, ending at $v$ and with $V(\gamma) \cap V(\rho) = \{u, v\}$, there exists a path $\lambda$ starting at $u$, ending at $v$ and with $V(\lambda) = V(\gamma) \cup V(\rho)$. A digraph $D$ is path-mergeable if it is $|V(D)|$-path-mergeable.

Definition 4.2. A path $\gamma = (x_0, \ldots, x_n)$ in a digraph $D$ is strongly internally non-augmentable if for every $0 \leq i < j \leq n$ there exists no path $\rho$ of length at least two, starting at $x_i$, ending at $x_j$ and with $V(\gamma) \cap V(\rho) = \{x_i, x_j\}$. We say that $\gamma$ is finally non-augmentable if there exists no $y \in V(D) - V(\gamma)$ such that $\overline{x_n y}.$

Proposition 4.3. Let $D$ be a path mergeable digraph. If $\gamma$ is a longest path in $D$, then $\gamma$ is strongly internally and finally non-augmentable.

Proof. Let $\gamma = (x_0, \ldots, x_n)$ be a longest path in $D$. Suppose that for $0 \leq i < j \leq n$ there is a path $\rho = (x_i, y_1, \ldots, y_r, x_j)$ with $r \geq 1$ and $V(\gamma) \cap V(\rho) = \{x_i, x_j\}$. If $j - i = 1$, then $(x_0, \ldots, x_i, y_1, \ldots, y_r, x_i+1, \ldots, x_n)$ is a path in $D$ of length $n + r$, contradicting that $\gamma$ is a longest path. Suppose that $j - i > 1$. Since $D$ is path-mergeable, there exists a path $\lambda$ starting at $x_i$, ending at $x_j$ and with $V(\lambda) = \{x_i, \ldots, x_j\} \cup V(\rho)$. Therefore there exists a path in $D$ of length $n + r$, namely $(x_0, \ldots, x_i-1)$ followed by $\lambda$ followed by $(x_{j+1}, \ldots, x_n)$, contradicting that $\gamma$ is a longest path. Finally, if there exists $y \in V(D) - V(\gamma)$ such that $\overline{x_n y}$, then $(x_0, \ldots, x_n, y)$ is a path in $D$ of length $n + 1$, contradicting that $\gamma$ is a longest path.

Lemma 4.4. Let $D$ be a strong digraph and let $\gamma = (x_0, \ldots, x_n)$ be a strongly internally and finally non-augmentable path in $D$. If $z \in V(D)$ is such that $\overline{zx_n}$, then $z \in V(\gamma)$.

Proof. Suppose that $z \notin V(\gamma)$. There exists a path $\rho = (y_0 = x_0, \ldots, y_m = z)$ starting at $x_0$ and ending at $z$ because $D$ is strong. Let $i_0 = \max\{i \mid x_i \in V(\rho)\}$ so that $y_{i_0} = x_{i_0}$ for some $j_0 < m$. If $i_0 < n$, then $\lambda = (y_{j_0}, \ldots, y_m, x_n)$ is a path in $D$ of length at least two, starting at $x_{i_0}$, ending at $x_n$ and with $V(\gamma) \cap V(\lambda) = \{x_{i_0}, x_n\}$, contradicting that $\gamma$ is strongly internally non-augmentable. If $i_0 = n$, then $\overline{x_n y_{j_0+1}}$, contradicting that $\gamma$ is finally non-augmentable because $y_{j_0+1} \notin V(\gamma)$.

Theorem 4.5. Let $D$ be a strong digraph. Then every maximal independent set intersects every strongly internally and finally non-augmentable path.

Proof. Let $\mathcal{I}$ be a maximal independent set. Suppose that $\gamma = (x_0, \ldots, x_n)$ is a strongly internally and finally non-augmentable path such that $V(\gamma) \cap \mathcal{I} = \emptyset$. There exists $z \in \mathcal{I}$ such that $\overline{zx_n}$ because $\mathcal{I}$ is a maximal independent set. Since $\gamma$ is finally non-augmentable, $\overline{zx_n}$, and hence, by lemma 4.4, $z \in V(\gamma)$, contradicting that $V(\gamma) \cap \mathcal{I} = \emptyset$. \qed
**Theorem 4.6.** If $D$ is a digraph, then there exists a maximal independent set that intersects every strongly internally and finally non-augmentable path.

**Proof.** We proceed by induction on the number of vertices. Clearly, the theorem is true if $|V(D)| = 1, 2$. Let $m > 2$ and suppose that the theorem is true for every digraph with $k < m$ vertices. Suppose that $|V(D)| = m$. If $D$ is strong, the result follows from theorem 4.5. Suppose that $D$ is not strong. Consider the acyclic condensation digraph $D^*$ that has a vertex for every maximal strong component of $D$, and for two vertices $u, v \in V(D^*)$, an arc from $u$ to $v$ if there exists an arc from a vertex in the corresponding component of $u$ to a vertex in the corresponding component of $v$. Since $D^*$ is acyclic, there exists $u_0 \in V(D^*)$ with $in(u_0) = 0$. Let $D'$ be the digraph that results from $D$ by removing the vertices in the component $C_0$ corresponding to $u_0$. Since $D$ is not strong, $D'$ is not the empty digraph and hence $|V(D')| < m$. Therefore there exists an independent set $\mathcal{I}' \subset V(D')$ that intersects every strongly internally and finally non-augmentable path in $D'$. Let $\mathcal{I} \subset V(D)$ be a maximal independent set in $D$ containing $\mathcal{I}'$, say $\mathcal{I} = \mathcal{I}' \cup \mathcal{I}_0$ for some $\mathcal{I}_0 \subset V(C_0)$. Let $\gamma = (x_0, \ldots, x_n)$ be a strongly internally and finally non-augmentable path in $D$. If $V(\gamma) \cap V(C_0) = \emptyset$, then $\gamma$ is a strongly internally and finally non-augmentable path in $D'$, and therefore $V(\gamma) \cap \mathcal{I} \neq \emptyset$. Suppose that $V(\gamma) \cap \mathcal{I} = \emptyset$. Then $V(\gamma) \cap V(C_0)$ and hence $x_0 \in V(C_0)$ because $in(u_0) = 0$. Actually, $V(\gamma) \subset V(C_0)$ because otherwise, if $i_0 = \min\{i \mid x_i \notin V(C_0)\}$, then $\gamma' = (x_{i_0}, \ldots, x_n)$ is a strongly internally and finally non-augmentable path in $D'$, and therefore $V(\gamma') \cap \mathcal{I}' \neq \emptyset$, contradicting that $V(\gamma') \cap \mathcal{I} = \emptyset$. There exists $z \in \mathcal{I}$ such that $\overline{x_nz}$ because $\mathcal{I}$ is a maximal independent set. Since $\gamma$ is finally non-augmentable, $\overline{zx_n}$ and therefore $z \in V(C_0)$, that is, $z \in \mathcal{I}_0$. We have a strong digraph $C_0$, a strongly internally and finally non-augmentable path $\gamma' \subset C_0$ and $z \in V(C_0)$ such that $\overline{zx_n}$. By lemma 4.4, $z \in V(\gamma)$, contradicting that $V(\gamma) \cap \mathcal{I} = \emptyset$ because $z \in \mathcal{I}_0 \subset \mathcal{I}$. \square

## 5 Locally in and out semicomplete digraphs

**Definition 5.1.** Let $D$ be a digraph. For every $u \in V(D)$, the **in-neighborhood** and **out-neighborhood** of $u$ are the sets $N^-(u) = \{x \in V(D) \mid xu \}$ and $N^+(u) = \{y \in V(D) \mid uy\}$ respectively.

**Definition 5.2.** A digraph $D$ is **locally in-semicomplete** if for every $u \in V(D)$, the digraph induced by the in-neighborhood of $u$ is semicomplete. A **locally out-semicomplete** digraph is defined similarly.

**Theorem 5.3.** Let $D$ be a locally in-semicomplete digraph. Then every maximal independent set intersects every non-augmentable path in $D$.

**Proof.** Suppose that there exist a non-augmentable path $\gamma = (x_0, \ldots, x_n)$ and a maximal independent set $\mathcal{I}$ such that $\gamma \cap \mathcal{I} = \emptyset$. Since $\mathcal{I}$ is a maximal independent set, there exists $z \in \mathcal{I}$ such that $\overline{x_nz}$. If $\overline{xnz}$, then $(x_0, \ldots, x_n, z)$ is a path in $D$ contradicting that $\gamma$ is non-augmentable. Then $\overline{xnz}$, that is, $z \in N^-(x_n)$. Since $\overline{x_{n-1}x_n}$, $x_{n-1} \in N^-(x_n)$ and therefore $\overline{x_{n-1}z}$ because $D$...
is locally in-semicomplete. If \( \overrightarrow{x_{n-1}} \), then \((x_0, \ldots, x_{n-1}, z, x_n)\) is a path in \( D \) contradicting that \( \gamma \) is non-augmentable. Then \( \overrightarrow{zx_{n-1}} \), that is, \( z \in N^{-}(x_{n-1}) \).

Since \( \overrightarrow{x_{n-2}x_{n-1}} \), \( x_{n-2} \in N^{-}(x_{n-1}) \) and therefore \( \overrightarrow{zx_{n-2}} \) because \( D \) is locally in-semicomplete. If \( \overrightarrow{x_{n-2}} \), then \((x_0, \ldots, x_{n-2}, z, x_{n-1}, x_n)\) is a path in \( D \) contradicting that \( \gamma \) is non-augmentable. Then \( \overrightarrow{zx_{n-2}} \), that is, \( z \in N^{-}(x_{n-2}) \).

Continuing in this way we get that \( \overrightarrow{zx_0} \), but then \((z, x_0, \ldots, x_n)\) is a path in \( D \) contradicting that \( \gamma \) is non-augmentable.

Theorem 5.4. Let \( D \) be a locally out-semicomplete digraph. Then every maximal independent set intersects every non-augmentable path in \( D \).

Proof. The proof is similar to the proof of theorem 5.3.

6 Semicomplete \( k \)-partite digraphs

Definition 6.1. A digraph \( D \) is semicomplete \( k \)-partite if there exist disjoint independent sets \( V_1, \ldots, V_k \subseteq V(D) \) with \( V_1 \cup \ldots \cup V_k = V(D) \) and such that for every \( i \neq j \), if \( u \in V_i \) and \( v \in V_j \), then \( uv \).

Theorem 6.2. Let \( D \) be a semicomplete \( k \)-partite digraph. Then every maximal independent set intersects every non-augmentable path in \( D \).

Proof. Let \( V_1, \ldots, V_k \) be as in definition 6.1. Clearly, if \( I \) is a maximal independent set, then \( I = V_{i_0} \) for some \( i_0 \in \{1, \ldots, k\} \). Let \( \gamma \) be a non-augmentable path in \( D \) and suppose that \( V(\gamma) \cap I = \emptyset \). Then \( V(\gamma) \subseteq V(D) - V_{i_0} \). If \( z \in V_{i_0} \), then \( \overrightarrow{zx} \) for every \( u \in V(D) - V_{i_0} \). In particular, \( \overrightarrow{zx} \) for every \( x \in V(\gamma) \), contradicting, by lemma 3.3, that \( \gamma \) is non-augmentable.

References


