Some results on the structure of kernel-perfect and critical kernel imperfect digraphs

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Abstract

A kernel $N$ of a digraph $D$ is an independent set of vertices of $D$ such that for every $w \in V(D) - N$ there exists an arc from $w$ to $N$. The digraph $D$ is said to be a kernel-perfect digraph when every induced subdigraph of $D$ has a kernel. Minimal non kernel-perfect digraphs are called critical kernel imperfect digraphs. In this paper some new structural results concerning finite critical kernel imperfect digraphs are presented. Also we present new sufficient conditions for a finite or infinite digraph to have a kernel.

Key words: infinite digraph, kernel, semikernel, semikernel modulo $F$, kernel-perfect digraph, critical kernel imperfect digraph.

1 Introduction

For general concepts we refer the reader to [2].

Let $D$ be a digraph; $V(D)$ and $A(D)$ will denote the set of vertices and arcs of $D$ respectively. Often we shall write $u_1u_2$ instead of $(u_1, u_2)$. Let $S_1$, $S_2$ be subsets of $V(D)$. The arc $u_1u_2$ will be called an $S_2S_2$-arc whenever $u_1 \in S_1$ and $u_2 \in S_2$; $D[S_1]$ will denote the subdigraph of $D$ induced by $S_1$. A set $I \subseteq V(D)$ is independent if $A(D[I]) = \emptyset$. A kernel $N$ of $D$ is and independent set of vertices such that for each $z \in V(D) - N$ there exists a $zN$-arc in $D$. A digraph $D$ is said to be kernel-perfect whenever every induced subdigraph of

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D has a kernel. And D is a critical kernel imperfect digraph when D has no kernel but every proper induced subdigraph of D has a kernel.

We write $F^-(S_i)$ (resp. $F^+(S_i)$) instead of $A(D[V(D), S_i])$ (resp. $A(D[S_i, V(D)])$ and $F^- u$, $F^+ u$ for $F^- \{u\}$, $F^+ \{u\}$) resp. If $D_0$ is a subdigraph (resp. induced subdigraph) of $D$, we write $D_0 \subset D$ (resp. $D_0 \subset^* D$). Let $D$ be a digraph and $C = (x_0, x_1, \ldots, x_{n-1}, x_0)$ (resp. $T = (x_0, x_1, \ldots, x_n)$) a directed cycle (resp. a directed path) contained in $D$; a pseudodiagonal of $C$ (resp. of $T$) is an arc $(x_i, x_j) \in (A(D) - A(C))$ (resp. $(x_i, x_j) \in (A(D) - A(T))$).

The concept of kernel of a digraph was introduced by J. Von Neumann and Morgenstern [13] in the context of Game Theory, and it has found many applications, for instance in cooperative n-person games, Nim-type games [2], in logic [1], and recently in computational complexity, artificial intelligence, combinatorics and coding theory. The problem of the existence of a kernel in a given digraph has been studied by several authors see by example [3] [4], [5], [6], [14]. The existence of kernels in infinite digraphs has been recently studied by some authors by example [9], [12], [15].

In this paper we present new sufficient conditions for a finite or infinite digraph to have a kernel. As a consequence some new structural results concerning finite critical kernel imperfect digraphs are obtained. At moment we do not know if there exists or does not an infinite critical kernel imperfect digraph.

**Definition 1 (Neumann-Lara, [12])** Let $D$ be a digraph. A semikernel $S$ of $D$ is an independent set of vertices such that for every $z \in V(D) - S$ for which there exists an $(S, z)$-arc there also exists a $(z, S)$-arc.

**Lemma 2** [12] Let $S$ be a semikernel of $D$, $B = \{v \in V(D) - S | \not\exists vS$-arc in $D\}$, and $S'$ a kernel of $D[B]$. Then $S \cup S'$ is a kernel of $D$.

**Definition 3 (Galeana-Sánchez, [8])** Let $D$ be a digraph and $F$ a set of arcs of $D$, a set $S \subseteq V(D)$ is called a semikernel modulo $F$ of $D$ if $S$ is an independent set of vertices such that, for every $z \in V(D) - S$ for which there exists an $(S, z)$-arc of $D - F$ there also exists a $(z, S)$-arc in $D$.

Along the paper $D_1$ will be a spanning subdigraph of $D$ and $D_2$ the subdigraph of $D$ defined as follows: $V(D_2) = V(D)$ and $A(D_2) = A(D) - A(D_1)$.

For $S \subseteq V(D)$, we denote by:

$$B_S = \{v \in V(D) - S | \text{there is no } (v, S)\text{-arc in } D\}$$

and when $D[B_S]$ has a nonempty semikernel modulo $A(D_1)$, say $S'$, we denote
by:
\[ T_{S'}^S = \{ v \in S \mid \text{there is no } (v, S')\text{-arc in } D_1 \} \]  \hspace{1cm} (2)

Also we denote by \( \alpha_{D_1} \) the set of nonempty semikernels modulo \( A(D_1) \) of \( D \).

**Definition 4** [10] We will say that a digraph \( D \) satisfies the **property** \( P(\alpha_{D_1}, \leq) \), whenever there exists a spanning subdigraph \( D_1 \) of \( D \) such that the following properties are fulfilled:

- There exists a partial order, \( \leq \), on the set of independent subsets of \( V(D) \).
- \( (\alpha_{D_1}, \leq) \) has a maximal element.
- For each \( S \in \alpha_{D_1} \), such that \( B_S \neq \emptyset \), (i.e. \( S \) is not a kernel), and each nonempty semikernel modulo \( A(D_1) \) of \( D[B_S] \), \( S' \), it satisfies that \( T_{S'}^S \cup S' \) is a nonempty semikernel modulo \( A(D_1) \) of \( D \) and \( T_{S'}^S \cup S' > S \).
- If \( S_0 \in \alpha_{D_1} \) is a maximal element, then for every \( S \in \alpha_{D_1} \) such that \( S < S_0 \) we have \( S \subset S_0 \cup \Gamma^-(S_0) \), \( \Gamma^-(S_0) = \{ u \in V(D) \mid \text{there exists a } (u, S_0)\text{-arc in } D \} \).

In [12] was proved that: If \( D \) is a digraph, \( D_1 \) the spanning subdigraph of \( D \) with \( A(D_1) = \emptyset \); and \( \leq \) is the partial order of the family of independent subsets of \( V(D) \) given by \( \subseteq \) (the contention). Then \( D \) satisfies \( P(\alpha_{D_1}, \leq) \).

Wide classes of digraphs satisfying the property \( P(\alpha_{D_1}, \leq) \) are constructed in [10], [11], [12] and [15].

If \( C = (u_0, u_1, \ldots, u_n, u_0) \) is a directed cycle, we denote by:
\[
C_{u_0}^0 = \{ u_i \mid i \equiv 0(\text{mod 2}); i \neq 0 \}; C_{u_0}^1 = \{ u_i \mid i \equiv 1(\text{mod 2}) \}; \]  \hspace{1cm} (3)

For a path \( P = (u_0, u_1, \ldots, u_n) \), we write:
\[
P^0 = \{ u_i \mid i \equiv 0(\text{mod 2}) \}; P^1 = \{ u_i \mid i \equiv 1(\text{mod 2}) \} \]  \hspace{1cm} (4)

### 2 Kernel-perfect Infinite Digraphs

Along this section the considered digraphs can be finite or infinite. In this section we introduce the concepts of semikernel and strong semikernel of a digraph modulo a set of vertices and a set of arcs and state Theorem 8 which is the main tool used in this paper.

**Definition 5** Let \( D \) be a digraph, \( A \subseteq A(D) \), \( I, R \subset V(D) \) and consider the following conditions:
i) $I \cap R^c$ is an independent set.

ii) $D$ does not contain $(I \cap R^c)I$-arcs.

iii) If $wv \in A(D) - A$, $u \in I \cap R^c$ and $v \in I^c \cap R^c$, then there exists $w \in I$ such that $vw \in A(D)$.

If conditions i) and ii) are satisfied, I will be called a semikernel of $D$ modulo $(R, A)$.

If conditions i'), (which is stronger that i)), and ii) are satisfied, I will be called a strong semikernel of $D$ modulo $(R, A)$.

**Definition 6** Let $D$ a digraph and $B$ a subdigraph of $D$. Suppose that $K \subset V(D)$. A directed path $T = (w_0, w_1, \ldots, w_n) \subseteq D$ will be called $(K, B)$-normal whenever $T$ satisfies:

i) $V(T) \cap K = \{w_j \mid 1 \leq j \leq n, j \text{ odd}\}$ or $V(T) \cap K = \{w_j \mid 0 \leq j \leq n, j \text{ even}\}$.

ii) If $s < j < n, w_j \in K^c, w_s \in K$, then $w_jw_s \notin A(D)$.

iii) If $w_j \in K$ then $w_jw_{j+1} \in Asim(D) \cap A(B)$.

**Remark 7** Notice that any $(K, B)$-normal directed path passes by $K$ and $K^c$ alternately and every arc, which begin in $K$, is in $Asim(D) \cap A(B)$.

**Theorem 8** Let $D$ be a digraph and $D_1$ an spanning subdigraph of $D$. If $I_0, I, R \subset V(D)$ are such that $\emptyset \neq I_0 \subset I$, $I_0 \cap R = \emptyset$ and satisfy

a) $I$ is a strong semikernel of $D$ modulo $(R, A(D_1))$.

b) every $(I, D_2)$-normal, $I_0R$-directed path passes by $U = \Gamma^-(I_0) \cap R^c$, then

$S = \{w \in I \mid \exists$ an $(I, D_2)$-normal, $I_0w$-directed path not passing by $U\}$ is a semikernel modulo $A(D_1)$ of $D$ which satisfies $I_0 \subset S \subset I \cap R^c$.

**Proof.** I) First we prove that $S$ is independent.

By a) $I_0$ is independent. By definition of $U$ and a),i) we have $U \subset I^c \cap R^c$. Since $I_0 \subseteq I$ and $U \subseteq I^c \cap R^c$ we have $I_0 \cap U = \emptyset$.

Now we prove $S \subseteq R^c$. Otherwise there exists $s \in S$ such that $s \in R$. Since $s \in S$ there exists an $(I, D_2)$-normal, $I_0s$-directed path $T$ not passing by $U$. Since $s \in R$, $T$ is an $(I, D_2)$-normal, $I_0R$-directed path. And by b), $T$ passes by $U$, which is a contradiction. Therefore $S \subseteq I \cap R^c$.

Now we conclude that $S$ is independent. Since $S \subseteq I \cap R^c$ and by a) i) $D$ contains no $(I \cap R^c)I$-arcs, particularly $D$ contains no SI-arcs (recall $S \subseteq I$). Therefore $S$ is an independent set.

II) For each $s \in S$ and $w \in V(D) - S$ such that $sw \in A(D_2)$ there is a $(w, S)$-arc in $D$. Let $s \in S$ and $w \in (V(D) - S)$ such that $(s, w) \in A(D_2)$. Since $s \in S$ we have there exists an $(I, D_2)$-normal, $I_0s$-directed path $T$, not passing by $U$ (by definition of $S$). Let $T = (w_0, w_1, \ldots, w_n = s)$ be, with $w_0 \in I_0$. We
extend \( T \) to the path \( T' = (w_0, w_1, \ldots, w_n, w) \), (see fig. 1)

\[
\begin{align*}
&\text{Observe that } l(T) \text{ is even and } T^0 \subset I. \\
&\text{(1)} \; w \in I^c. \text{ If } w \in I \text{ then we have an } (I \cap R^c)I\text{-arc, (as } s \in S \subseteq I \cap R^c) \\
&\text{contradicting that } I \text{ is a strong semikernel of } D \text{ modulo } (R, A(D_1)), (i') \\\n&\text{(2)} \; \text{We may assume that } T' \text{ is a path. Otherwise } w = w_i, \; i \text{ odd, } i < n, \\
&\text{therefore } w w_{i+1} \in A(D) \text{ with } w_{i+1} \in S, \text{ (notice that } (w_0, T, w_{i+1}) \text{ is an } \\
&(I, D_2)\text{-normal, } I_0w_{i+1}\text{-path not passing by } U, \text{ then } w_{i+1} \in S), \text{ and we } \\
&\text{are done.} \\
&\text{(3)} \; T' \text{ is } (I, D_2)\text{-normal. Remember that } w_nw \in A(D_2) \cap Asim(D), \text{ and } \\
&w \in I^c \text{ (by (1)).} \\
&\text{(4)} \; \text{We may assume that } w \notin R. \text{ Otherwise } T' \text{ is an } (I, D_2)\text{-normal, } I_0R\text{-path,} \\
&\text{not passing by } U, \text{ since } T \text{ do not pass by } U \text{ and } w \notin U \text{ (if } w \in U \text{ then we } \text{have a } wS\text{-arc as required, recall } I_0 \subseteq S), \text{ contradicting the hypothesis} \\
&\text{b).} \\
&\text{(5)} \; w \in I^c \cap R^c, \text{ by 1 and 4.} \\
&\text{(6)} \; \text{There exists } z \in I \text{ such that } wz \in A(D). \text{ Because } s \in S \subset I \cap R^c, w \in \\
&I^c \cap R^c, sw \in A(D_2) \text{ and } I \text{ is strong semikernel of } D \text{ modulo } (R, A(D_1)). \\
&\text{(7)} \; \text{We may assume that } T'' = (w_0, w_1, \ldots, w_n, w, z) \text{ is an } (I, D_2)\text{-normal path} \\
&\text{not passing by } U. \\
&\text{ (a)} \; T'' \text{ is a path. Otherwise } z = w_i, \; i \text{ even, } 0 \leq i \leq n \text{ and we have a} \\
&wS\text{-arc in } D \text{ and we are done.} \\
&\text{ (b)} \; T'' \text{ is } (I, D_2)\text{-normal. Recall } T' \text{ is also } (I, D_2)\text{-normal by (3) and} \\
&z \in I. \\
&\text{ (c)} \; T'' \text{ do not pass by } U. \text{ First; } T' \text{ do not pass by } U \text{ (because } T \text{ do not} \\
&\text{pass by } U \text{ and we may assume } w \notin U \text{ and } z \notin U \text{ (by (6), and } \\
&U \subset I^c). \\
\end{align*}
\]

Thus \( z \in S \) and \( wz \in A(D) \), therefore \( S \) is a semikernel modulo \( A(D_1) \) of \( D \). ■
Theorem 9 Let $D$ be a digraph and $D_1$ a spanning subdigraph of $D$. Suppose that $I_0, I, R \subset V(D)$ with $\emptyset \neq I_0 \subseteq I$ and $I_0 \cap R = \emptyset$, satisfy the following conditions:

i) $I$ is a strong semikernel of $D$ modulo $(R, A(D_1))$.

ii) $D$ contains no semikernel modulo $A(D_1)$, $S$, such that $I_0 \subset S \subset I \cap R^c$.

Then, there exists a direct $(I, D_2)$-normal, $I_0 R$-directed path $T = (t_0, \ldots, t_n)$, not passing by $U = \Gamma^{-1}(I_0) \cap R^c$, which satisfies the following properties:

a) $T$ has no $(V(T) - t_n)T^0$-pseudodiagonals.

b) $l(T)$ is even iff $t_n \in I$.

Proof. By Theorem 8, there exists an $(I, D_2)$-normal, $I_0 R$-directed path, not passing by $U$. Choose $T = (t_0, \ldots, t_n)$ so that $l(T)$ takes the minimum possible value.

1) $T^0 \subseteq I$. Due to $t_0 \in I_0 \subset I$ and $T$ is $(I, D_2)$-normal.

2) $V(T) \cap R = \{t_n\}$. Otherwise there exists $t_i$, $i < n$ and $t_i \in R$ then $T' = (t_0, T, t_i)$ is an $(I, D_2)$-normal, $I_0 R$-directed path, not passing by $U$ and $l(T') < l(T)$, contradicting the minimality of $T$.

3) $t_2, \in I \cap R^c$ and $t_{2i+1} \in I^c \cap R^c$, for all $0 \leq 2i < n$, $0 \leq 2j + 1 < n$, due to 2) and $T$ is $(I, D_2)$-normal.

4) $T^0 - t_n \subseteq I \cap R^c$ (follows from (1) and (3)).

5) $T$ contains no $(T^0 - t_n)T^0$-pseudodiagonals, because $I$ is a strong semikernel of $D$ modulo $(R, A(D_1))$, $(T^0 - t_n) \subset I \cap R^c$ and $T^0 \subseteq I$.

6) $T$ has no $(T^1 - t_n)T^0$-pseudodiagonals. Otherwise there exist $t_{2i+1}$ and $t_{2j}$, $0 < 2i + 1 < n$, $0 \leq 2j \leq n$, such that $t_{2i+1}t_{2j} \in A(D)$.

Case 1) $2i + 1 < 2j$. then $(t_0, T, t_{2i+1}) \cup (t_{2i+1}t_{2j}) \cup (t_{2j}, T, t_n)$ is an $(I, D_2)$-normal $I_0 R$-directed path, not passing by $U$, whose length is smaller than $l(T)$, which is a contradiction again.

Case 2) $2i + 1 > 2j$, this case is impossible as $T$ is $(I, D_2)$-normal.

7) $l(T)$ is even iff $t_n \in I$. If $l(T)$ is even then $n$ is even. From the definition of $T_0$ we have $t_n \in T^0$ and from (1) $t_n \in I$. Conversely if $t_n \in I$ then $t_{n-1} \in I^c$ (as $T$ is $(I, D_2)$-normal). From (3) we have $n - 1$ is odd. Thus $n = l(T)$ is even.

$\blacksquare$

Definition 10 Let $D$ be a digraph and $D_1$ a spanning subdigraph of $D; D$ is said to be kernel-perfect modulo $A(D_1)$, if every induced subdigraph of $D$ has a nonempty semikernel modulo $A(D_1)$.

Theorem 11 Let $D$ be a digraph that satisfies $P(\alpha_{D_1}, \leq)$; and $I_0, I, R \subset V(D)$ such that $\emptyset \neq I_0 \subset I$, $I_0 \cap R = \emptyset$. Suppose that conditions i), ii) and iii) are satisfied:

i) $I$ is a strong semikernel of $D$ modulo $(R, A(D_1))$.

ii) $D$ has no kernel.
iii) $D - I_0$ is a kernel-perfect modulo $A(D_1)$ digraph.

Then there exists a direct $(I, D_2)$-normal, $I_0R$-directed path $T = (t_0, t_1, \ldots, t_n)$, not passing by $\Gamma^-(I_0) \cap R^c$, which satisfies: a) $T$ has no $(V(T) - t_n)T^0$-pseudodiagonals and b) $l(T)$ is even iff $t_n \in I$.

**Proof.** Suppose that the Theorem 11 is false, then (by Teo. 9) $D$ has a semikernel modulo $A(D_1)$, $S \neq \emptyset$, such that $I_0 \subset S \subset I \cap R^c$. Since $D$ satisfies $P(\alpha_{D_1}, \leq)$, then $(\alpha_{D_1}, \leq)$ has a maximal element.

Case 1) $S$ is a maximal element of $(\alpha_{D_1}, \leq)$. By hypothesis $S$ is not a kernel of $D$, then $B_S \neq \emptyset$. Since $I_0 \subset S$, $B_S \subset D - I_0$, and $D - I_0$ is a kernel perfect modulo $A(D_1)$ digraph; we have $D[B_S]$ is a kernel-perfect modulo $A(D_1)$ digraph. Let $S'$ be a nonempty semikernel modulo $A(D_1)$ of $D[B_S]$. Therefore $T_S' \cup S'$ is a nonempty semikernel modulo $A(D_1)$ of $D$ and $T_S' \cup S' > S$, (as $D$ satisfies $P(\alpha_{D_1}, \leq)$), contradicting that $S$ is a maximal element of $(\alpha_{D_1}, \leq)$.

Case 2) $S$ is not maximal element of $(\alpha_{D_1}, \leq)$, then there exists a maximal element of $(\alpha_{D_1}, \leq)$ say $S_0$, such that $S_0 > S$. By $ii)$, $S_0$ is not a kernel, therefore $B_{S_0} \neq \emptyset$.

By the definition of $B_{S_0}$, we have that $B_{S_0} \subset D - S_0$ and $B_{S_0} \cap \Gamma^-(S_0) = \emptyset$. So $B_{S_0} \subset D - (S_0 \cup \Gamma^-(S_0))$. Since $D$ satisfies $P(\alpha_{D_1}, \leq)$, $S_0$ is a maximal element of $(\alpha_{D_1}, \leq)$ and $S < S_0$, it follows that $S \subset S_0 \cup \Gamma^-(S_0)$. Thus $I_0 \subset S \subset S_0 \cup \Gamma^-(S_0)$ and $D - (S_0 \cup \Gamma^-(S_0)) \subset D - I_0$. And then $B_{S_0} \subset D - I_0$.

Hence $iii)$ implies $D[B_{S_0}]$ has a semikernel modulo $A(D_1)$, say $S'$. Finally we have: $T_{S_0} \cup S'$ is a semikernel modulo $A(D_1)$ of $D$, such that $T_{S_0} \cup S' > S_0$, (as $D$ satisfies $P(\alpha_{D_1}, \leq)$) contradicting the maximality of $S_0$. $\blacksquare$

3 Structure of some digraphs without kernel

The digraphs considered in this section can be finite or infinite.

**Theorem 12** Let $D$ be a digraph that satisfies $P(\alpha_{D_1}, \leq)$, $u \in V(D)$, $N_u$ a kernel of $D - u$. Suppose that conditions i) to iii) are satisfied:

i) $D - v$ is kernel-perfect modulo $A(D_1)$.

ii) $D - w$ is kernel-perfect modulo $A(D_1)$, for every $w \in \Gamma^+(v)$.

iii) $D$ has no kernel.

Then there exists an $(N_u, D_2)$-normal, $vu$-directed path $T$ without $(V(T) - u)T^i$-pseudodiagonals (where $i$ is the residue of $l(T) + 1$ modulo 2).

**Proof.** Take $I = N_u$, $R = \{u\}$ and define $I_0$ as follows: If $v \in N_u$, $I_0 = \{v\}$; otherwise $I_0 = \Gamma^+(v) \cap N_u$. Since the conditions of Theorem 11 are fulfilled, Theorem 12 follows. $\blacksquare$
Definition 13 Let \( C = (u_0, u_1, \ldots, u_n, u_0) \) (resp. \( T = (u_0, u_1, \ldots, u_n) \)) be a directed cycle (resp. a directed path). For a fixed \( i \in \{1, 2\} \) we will say that \( C \) (resp. \( T \)) is \((C^0, D_2)\)-normal (resp. \((T^1, D_2)\)-normal) if every arc of \( C \) (resp. of \( T \)), which starts in \( C^0 \) (resp. \( T^1 \)), is in \( A(D_2) \).

Corollary 14 Let \( D \) be a digraph that satisfies the property \( P(\alpha_{D_1}, \lessdot) \) and \( f = uv \in A(D) \). Suppose that \( D \) has no kernel and satisfies:

i) \( D - u \) has kernel.

ii) \( D - v \) is kernel-perfect modulo \( A(D_1) \).

iii) \( D - w \) is kernel-perfect modulo \( A(D_1) \), for every \( w \in \Gamma^+(v) \).

Then there exists a \((C^0, D_2)\)-normal directed cycle \( C \) of odd length passing by \( f \) and having no \( V(C)(C^0_u)\)-pseudodiagonals (In particular \( C^0_u \) is independent).

Proof. Let \( N_u \) be a kernel of \( D - u \). By Theorem 12, there exists an \((N_u, D_2)\)-normal \( vu \)-directed path \( T \) without \((V(T) - \{u\})T^1\)-pseudodiagonals (where \( i \) is the residue of \( l(T) + 1 \) modulo \( 2 \)). Since \( N_u \) is a kernel of \( D - u \), \( D \) has no kernel and \( uv \in A(D) \) we have \( v \notin N_u \). Therefore \( l(T) \) is even (as \( \{u, v\} \subseteq N^c_u \) and \( T \) is \((N_u, D_2)\)-normal). Thus \( T \) has no \((V(T) - \{u\})T^1\)-pseudodiagonals. Therefore \( C = uv \cup T \) is an odd directed cycle \((C^0_u, D_2)\)-normal passing by \( f \) and having no \( V(C)(C^0_u)\)-pseudodiagonals (notice that \( T^1 = C^0_u \)). (Notice that \( C \) has no \( uC^0_u\)-pseudodiagonals; otherwise \( N_u \) is a kernel of \( D \), as \( T^1 = C^0_u \subseteq N_u \).)

Theorem 15 Let \( D \) be a digraph, which satisfies \( P(\alpha_{D_1}, \lessdot) \) and \( u \in V(D) \). If \( \emptyset \neq A \subset F^+_{u} \) and \( I_0 = \{z \in V(D) \mid uz \in A\} \) satisfy:

i) \( D - A \) has a kernel but \( D - A' \) has no kernel for \( A' \subsetneq A \).

ii) \( D - I_0 \) is kernel-perfect modulo \( A(D_1) \).

Then there exist \( f \in A \) and a \((C^1, D_2)\)-normal, directed cycle \( C \) of odd length passing by \( f \), not intersecting \( \Gamma^-(I_0) - \{u\} \) and without \( V(C)(C^1_u \cup \{u\})\)-pseudodiagonals.

Proof. By i) \( D \) has no kernel. Let \( I \) be a kernel of \( D - A \) and \( R = \{u\} \). Clearly \( I \) is a strong semikernel of \( D \) modulo \( (\{u\}, D_1) \). From i) we have \( I_0 \cup \{u\} \subset I \).

By Theorem 11 there exists a \((I, D_2)\)-normal, \( I_0u \)-directed path not passing by \( \Gamma^-(I_0) \cap \{u\} \) which satisfies:

a) \( T \) has no \((V(T) - t_n)T^0\)-pseudodiagonals.

b) \( l(T) \) is even iff \( t_n \in I \).

Let be \( T = (z = w_0, w_1, \ldots, w_n = u) \). Notice that \( l(T) \) is even because \( I_0 \cup \{u\} \subset I \) and \( T \) is \((I, D_2)\)-normal. Adding the arc \( f = uz \) to \( T \) we get a directed cycle \( C \); clearly \( T^0 = (C^1_u \cup \{u\}) \) and \( C \) satisfies the required properties. Notice that we may assume that \( C \) has no \( uC^1_u\)-pseudodiagonals. Otherwise we take \( m = \max \{i \in \{0, \ldots, n\} \cap C^1_u \mid such \, that \, uw_i \in A(D)\} \) and clearly \( C^1 = uw_i \cup (w_i, T, u) \) satisfies the required properties.

Corollary 16 Let \( D \) be a digraph, which satisfies \( P(\alpha_{D_1}, \lessdot) \) and \( f = uv \in
Corollary 17 Let $D$ be a digraph, which satisfies $P(\alpha_{D_1}, \leq)$ and $f = uv \in A(D)$. If $D$ does not have a kernel and $D - f$ is kernel-perfect, then there exists a $(\mathcal{C}_D, D_2)$-normal, directed cycle $\mathcal{C}$, of odd length containing $f$, not intersecting $\Gamma^-(I_0) - \{u\}$ and without $V(\mathcal{C})(\mathcal{C}_D \cup \{u\})$-pseudodiagonals.

Proof. Clearly we have that $D - f$ is kernel perfect modulo $A(D_1)$ as $f = uv$ and $D - f$ is kernel perfect. ■

Corollary 18 Let $D$ be a digraph which satisfies $P(\alpha_{D_1}, \leq)$ and $u \in V(D)$. If $D$ has no kernel, $D - u$ is kernel-perfect and $D - w$ is kernel-perfect modulo $A(D_1)$ for all $w \in \Gamma^+(u)$, then there exists a $(\mathcal{C}_D, D_2)$-normal, directed cycle $\mathcal{C}$ of odd length passing by $u$ and without $V(\mathcal{C})(\mathcal{C}_D \cup \{u\})$-pseudodiagonals.

Proof. First we prove that $D - F_u^+$ possesses a kernel which contains $\{u\}$. Clearly $\{u\}$ is a nonempty semikernel of $D - F_u^+$. Consider the set $B_u = \{v \in (V(D) - F_u^+) - \{u\} \mid \text{\#vu-arc in } D - F_u^+\}$. If $B_u \neq \emptyset$ then $\{u\}$ is a kernel with the required properties. Otherwise $(D - F_u^+)[B_u]$ has a kernel, say $S'$ (as $D - u$ is kernel-perfect and $B_u \subseteq V(D) - \{u\}$). Therefore from Lemma 2 we have that $\{u\} \cup S'$ is a kernel of $D - F_u^+$. Take $N_u$ kernel of $D - F_u^+$ containing $u$, such that $|A(D[N_u]) \cap F_u^+|$ takes the minimum possible value. Let be $A = F_u^+ \cap A(D[N_u])$. And apply the Theorem 15. ■

Theorem 19 Let $D$ be a digraph, which satisfies $P(\alpha_{D_1}, \leq)$ and $u \in V(D)$. If $\emptyset \neq A \subset F_u^+ \cup A$ has the following properties:

i) $D - A$ has a kernel.
ii) $D - A'$ has no kernel for $A' \subset A$.  
iii) $D - u$ is kernel-perfect modulo $A(D_1)$.

Then there exist $f = wu \in A$ and a $(\mathcal{C}_w, D_2)$-normal, directed cycle $\mathcal{C}$ of odd length passing by $f$, not intersecting $\Gamma^-(u) - w$ and without $V(\mathcal{C})(\mathcal{C}_w \cup \{w\})$-pseudodiagonals.

Proof. Let $I$ be a kernel of $D - A$ and take $I_0 = \{u\}$ and $R = \{z \in V(D) \mid zu \in A\}$. By ii) $R \cup \{u\} \subset I$. By Theorem 11, there exists a direct $(I, D_2)$-normal, $wR$-directed path $T = (u = t_0, t_1, \ldots, t_n = w)$ not passing by $\Gamma^-(u) \cap R^c$ such that: $T$ has no $(V(T) - \{t_n\})T^0$-pseudodiagonals. And $l(T)$ is even iff $w \in I$. Then $l(T)$ is even (as $w \in I$). Adding $f = wu$ to $T$, we obtain an odd directed cycle $\mathcal{C}$. And $\mathcal{C}$ has the required properties as $V(T) = V(\mathcal{C})$ and $T^0 = \mathcal{C}_w \cup \{w\}$. (Notice that: since $w = t_n \in I$, $\mathcal{C}_w \subseteq I$ and $I$ is a kernel of $D - A$; we have that $\mathcal{C}$ has no $t_n(\mathcal{C}_w \cup \{w\})$-pseudodiagonals). ■

Corollary 20 Let $D$ be a digraph that satisfies $P(\alpha_{D_1}, \leq)$ and $u \in V(D)$. If
D has no kernel and $D - u$ is kernel-perfect, then there exists $f = vu \in A(D)$ and an odd $(\mathcal{C}_v^1, D_2)$-normal directed cycle $\mathcal{C}$, passing by $f$, not intersecting $\Gamma^-(u) - v$ and without $V(\mathcal{C})(\mathcal{C}_v^1 \cup \{v\})$-pseudodiagonals.

**Proof.** Let $N_u$ be a kernel of $D - u$ such that $|N_u \cap \Gamma^-(u)|$ takes the minimum possible value. Take $A = \{vu \mid v \in N_u \cap \Gamma^-(u)\}$ and apply Theorem 19. (Notice that since $D$ has no kernel, we have $N_u \cup \{u\}$ is a kernel of $D - A$.)

4 Critical kernel imperfect digraphs’ structure

Along this section the concerning digraphs will be finite. As at the moment we do not know if there exists or does not an infinite critical kernel imperfect digraph.

**Theorem 21** Let $D$ be a critical kernel imperfect digraph that satisfies $P(\alpha_{D_1}, \preceq)$ and $u, v \in V(D)$, then there exists a $(T^i, D_2)$-normal, $vu$-directed path, $T$, having no $V(T)T^i$-pseudodiagonals. (where $i$ is the residue $l(T) + 1 \mod 2$).

**Proof.** It follows directly from Theorem 12.

**Corollary 22** A critical kernel imperfect digraph $D$ that satisfies $P(\alpha_{D_1}, \preceq)$ is strongly connected.

**Corollary 23** (Duchet, [3]) A finite critical kernel imperfect digraph $D$ is strongly connected.

**Theorem 24** Let $D$ be a critical kernel imperfect digraph that satisfies $P(\alpha_{D_1}, \preceq)$ and $f = uv \in A(D)$. Then there exists a $(\mathcal{C}_u^0, D_2)$-normal, directed cycle $\mathcal{C}$, of odd length containing $f$ and having no $V(\mathcal{C})\mathcal{C}_u^0$-pseudodiagonals.

**Proof.** It follows directly from Corollary 14.

**Corollary 25** Let $D$ be a critical kernel imperfect digraph that satisfies $P(\alpha_{D_1}, \preceq)$ and $u \in V(D)$. Then there exists a $(\mathcal{C}_u^0, D_2)$-normal, directed cycle $\mathcal{C}$, of odd length passing by $u$, which contains neither $V(\mathcal{C})\mathcal{C}_u^0$-pseudodiagonals, neither $u\mathcal{C}_u^0$-pseudodiagonals.

**Proof.** It follows from Theorem 24 that there exists a directed cycle $\mathcal{C} = (u = u_0, u_1, \ldots, u_n, u_0)$ passing by $u$ which has no $V(\mathcal{C})\mathcal{C}_u^0$-pseudodiagonals. If $\mathcal{C}$ possesses $u\mathcal{C}_u^1$-pseudodiagonals the we take $i_0 = \max\{i \in V(\mathcal{C}) - \mathcal{C}_u^0 \text{ such that } uu_i \text{ is a pseudodiagonal of } \mathcal{C}\}$. Clearly $\mathcal{C}' = (u, u_{i_0}) \cup (u_{i_0}, \mathcal{C}, u)$ satisfies the required properties.
Theorem 26 Let $D$ be critical kernel imperfect digraph which satisfies $P(\alpha_{D_1}, \leq)$ and $u \in V(D)$. Then there exists a ($C^1_u, D_2$)-normal, directed cycle, $C$, of odd length passing by $u$ and having no $V(C)(C^1_u \cup \{u\})$-pseudodiagonals.

Proof. It follows directly from Corollary 25 and Theorem 26 we there exist

Theorem 27 Let $D$ be a critical kernel imperfect digraph that satisfies $P(\alpha_{D_1}, \leq)$ and $u \in V(D)$, then for some $f = vu \in A(D)$ there exists a ($C^1_v, D_2$)-normal, directed cycle of odd length passing by $f$, having no $V(C)(C^1_v \cup \{v\})$-pseudodiagonals which do not intersect $\Gamma^-(u) - v$.

Proof. It follows directly from Corollary 18.

Definition 28 Let $D$ be a digraph, $C = (u_0, u_1, \ldots, u_{2m}, u_0)$ an odd directed cycle of $D$ and $F$ a subdigraph of $D$. We will say that $C$ is weakly $F$-alternated, if $C$ possesses $m$ arcs in $A$ appearing alternated in $C$, that means, there exists an arc $f = u_iu_{i+1} \in A(F) \cap A(C)$, such that $C$ is ($C^0_{i+1}, F$)-normal.

A pole of a $C$ is the final vertex of a pseudodiagonal of $C$.

Claim 1 If $C = (u = u_0, u_1, \ldots, u_{2n}, u_0)$ is an odd directed cycle such that: $C$ is ($C^1_u, D_2$)-normal, for some $i \in \{1, 0\}$, and $C$ has no $V(C)(C^1_v \cup \{v\})$-pseudodiagonals or $C$ has no $V(C)(C^1_v \cup \{v\})$-pseudodiagonals, then $C$ is a weakly $D_2$-alternated cycle which has no two consecutive poles.

Theorem 29 Let $D$ be a critical kernel imperfect digraph that satisfies $P(\alpha_{D_1}, \leq)$ which is not a directed cycle of odd length, and $u \in V(D)$. Then there exist $f' \in F_u^-$ and $f'' \in F_u^+$, such that each one of them belongs to at least two different weakly $D_2$-alternated cycles of odd length, having no two consecutive poles.

Proof. By Theorem 27 there exists a ($C^1_u, D_2$)-normal, directed cycle, $C = (w = w_0, u = w_1, w_2, \ldots, w_{2n}, w_0)$, of odd length passing by some $f' = wu$ and having no $V(C)(C^1_u \cup \{v\})$-pseudodiagonals. By the Claim 1, $C$ is weakly $D_2$-alternated and has no two consecutive poles. From Theorem 24 there exists a ($C^0_{u_0}, D_2$)-normal directed cycle of odd length containing $f' = (w_0, u)$ and having no $V(C')(C^0_{u_0} \cup \{v\})$-pseudodiagonals. Let $\bar{C} = (w = w_0, u = \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_{2n}, \bar{w}_0)$ be such a cycle. From Claim 1 we have that $C$ is weakly $D_2$-alternated and without two consecutive poles. Moreover $\bar{C} \neq C$, for otherwise $C = \bar{C}$ would be an induced subdigraph of $D$ and then $D[V(C)] = C$ contradicting our hypothesis. In a similar way and applying Corollary 25 and Theorem 26 we prove the existence of $f''$.

Corollary 30 Let $D$ be a critical kernel imperfect digraph that satisfies $P(\alpha_{D_1}, \leq)$ which is not a directed cycle of odd length and $u \in V(D)$. Then $u$ belongs to at least $\Delta_D(u) + 1$ weakly $D_2$-alternated directed cycles of odd
length and without two consecutives poles ($\Delta_D(u) = \max\{|\Gamma^-(u)|, |\Gamma^+(u)|\}$).

Proof. It follows directly from: Theorem 29; Theorem 24 and Claim 1.

5 Kernel-perfect digraphs.

In this section we study some sufficient conditions for a finite digraph to be kernel-perfect.

Definition 31 We say that $D$ satisfies hereditarily $P(\alpha_{D_1}, \leq)$, if $D$ satisfies $P(\alpha_{D_1}, \leq)$ and for every $F \subset V(D)$, $D[F]$ satisfies $P(\alpha_{D_1[F]}, \leq)$, where $\leq$ is the restriction to $\alpha_{D_1[V(F)]}$ of $\leq$. (Notice that every independent set of $D[F]$ is also an independent subset of $V(D)$.

Theorem 32 Let $D$ be a finite digraph that satisfies hereditarily $P(\alpha_{D_1}, \leq)$ and $T \subset V(D)$, such that $D - T$ is kernel-perfect. Furthermore suppose that for every $u \in T$ either a) or b) is satisfied:

a) Every $((\mathcal{C}_0^0, D_2^0)$-normal, directed cycle, $\mathcal{C}$, of odd length passing by $u$ has at least one $V(\mathcal{C})\mathcal{C}_0^0$-pseudodiagonal.

b) Every $((\mathcal{C}_1^1, D_2^1)$-normal, directed cycle, $\mathcal{C}$, of odd length passing by $u$ has at least one $V(\mathcal{C})(\mathcal{C}_1^1 \cup \{u\})$-pseudodiagonal.

Then $D$ is kernel-perfect.

Proof. If $D$ is not a kernel-perfect digraph, then $D$ contains an induced critical kernel imperfect subdigraph $H$. Since $D - T$ is kernel-perfect, $V(H) \cap T \neq \emptyset$. Take any $u \in V(H) \cap V(T)$. By Theorem 24 (resp. 26), there exists a $((\mathcal{C}_1^0, D_2^1)$-normal, directed cycle, $\mathcal{C}_1$, of odd length, (with $D_2 = H - D_1$) without $V(\mathcal{C}_1)\mathcal{C}_1^0$-pseudodionals (resp. a $((\mathcal{C}_1^0, D_2^1)$-normal), directed cycle, $\mathcal{C}_2$, of odd length without $V(\mathcal{C}_2)((\mathcal{C}_1^1 \cup \{u\})$-pseudodionals passing by $u$. Thus neither a) nor b) is satisfied in $H$ and consequently in $D$. Therefore the hypothesis is thus contradicted.

Theorem 33 Let $D$ be a finite digraph that satisfies hereditarily $P(\alpha_{D_1}, \leq)$ and $A \subset A(D)$. Suppose that every $f = uv \in A$ satisfies:

i) Each $((\mathcal{C}_0^0, D_2^1)$-normal, directed cycle, $\mathcal{C}$, of odd length passing by $f$ has some $V(\mathcal{C})\mathcal{C}_0^0$-pseudodiagonal.

Then $D$ is kernel-perfect if and only if every induced subdigraph $H$ of $D$ such that $A(H) \cap A = \emptyset$ is kernel-perfect.

Proof. If $D$ is not a kernel-perfect digraph, then $D$ contains an induced critical kernel imperfect digraph $H$. It follows by hypothesis that $A(H) \cap A \neq \emptyset$. Take $f = uv \in A(H) \cap A$ and apply Theorem 24; condition i) is thus contradicted. The converse is obvious.
Let $C$ be a directed cycle of odd length and $\mathcal{P}(C) = \{w \in V(C) \mid \exists V(C)w -$ pseudodiagonal of $C\}$. We denote by:

$$C^{(1)} = \bigcup_{v \in \mathcal{P}(C)} C_v^1,$$

$$C^{(0)} = \mathcal{P}(C) \cup \bigcup_{u \in \mathcal{P}(C)} C_u^0$$

**Corollary 34** Let $D$ be a finite digraph that satisfies hereditarily $P(\alpha_{D_1}, \leq)$ and $T \subset V(D)$. Suppose that $D - T$ is kernel-perfect. If every weakly $D_2$-alternated, directed cycle, $C$, of odd length such that $V(C) \cap T \neq \emptyset$, satisfies $C = C^{(0)}$, then $D$ is kernel-perfect.

**Proof.** If $D$ is not a kernel-perfect digraph, then by Theorem 32, there exists $u \in T$, such that neither a) nor b) are satisfied. Then there exists a $(C_u^1, D_2)$-normal, directed cycle, $C$, of odd length, weakly $D_2$-alternated and without $V(C)(C_u^1 \cup \{u\})$-pseudodiagonal with $u \in V(C)$. On the other hand we have from the hypothesis that $C = C^{(0)}$. Thus there exists $w \in \mathcal{P}(C)$ such that $u \in C_u^0$ (as $u \notin \mathcal{P}(C)$). Therefore $w \in C_u^1 \cap \mathcal{P}(C)$. Contradicting that $C$ has no $V(C)(C_u^1 \cup \{u\})$-pseudodiagonals. ■

**Corollary 35** Let $D$ be a finite digraph that satisfies hereditarily $P(\alpha_{D_1}, \leq)$ and $A \subset A(D)$. Suppose that every weakly $D_2$-alternated directed cycle, $C$, of odd length such that $A(C) \cap A \neq \emptyset$, satisfies $C = C^{(1)}$. Then: $D$ is kernel-perfect if and only if every induced subdigraph $H$ of $D$ such that $A(H) \cap A = \emptyset$ is kernel-perfect.

**Proof.** Let $f = uw \in A$ be and $C$ a weakly $D_2$-alternated $(C_u^0, D_2)$-normal, directed cycle of odd length passing by $f$. We will prove that $C$ has a $V(C)C_u^0$-pseudodiagonal. By hypothesis, $C = C^{(1)}$, therefore $u \in C_u^1$, for some $w \in \mathcal{P}(C)$. It follows $w \in C_u^0 \cap \mathcal{P}(C)$. Thus $C$ has a $V(C)C_u^0$-pseudodiagonal. Hence the hypothesis of Theorem 33 are fulfilled and the Corollary 35 follows from Theorem 33. ■

**Remark 36** [7] Let $C = (u_0, u_1, \ldots, u_{2n}, u_0)$ be a directed cycle in $D$ of odd length, $\mathcal{P}(C) = \{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}; 0 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq 2n$. Then

1. $V(C) = C^{(0)}$ if and only if:
   i.1) there exists $j$, $1 \leq j \leq k$, such that $i_{j+1} = i_j + 1$.
   or
   i.2) there exists $j, k, l$, $1 \leq j < l \leq k$, such that both, the $u_iu_{i+1}$-directed path and the $u_{i_1}u_{i+1}$-directed path contained in $C$, have odd length (addition is taken mod $k$).
2. $V(C) = C^{(1)}$ if and only if i.2).
Proposition 37 Let $D$ be a finite digraph which satisfies hereditarily $P(\alpha_{D_1}, \preceq)$ and $T \subset V(D)$. Suppose that $D - T$ is a kernel-perfect and for every weakly $D_2-$alternated, directed cycle, $C = (u_0, u_1, \ldots, u_{2n}, u_0)$, of odd length, such that $V(C) \cap T \neq \emptyset$, there exists $i$ such that $u_i, u_{i+1} \in P(C)$. Then $D$ is a kernel-perfect.

Proof. Notice that $C = C^{(0)}$ and apply Corollary 34. □

Proposition 38 Let $D$ be a finite digraph which satisfies hereditarily $P(\alpha_{D_1}, \preceq)$. If every weakly $D_2-$alternated, directed cycle, $C$, of odd length such that for some $uv \in A(C)$, $vu \notin A(D)$, has a pseudodiagonal $f_c$, such that for each weakly $D_2-$alternated, directed cycle $\gamma$ of odd length containing $f_c$, $\gamma = \gamma^{(1)}$, then $D$ is kernel-perfect.

Proof. Assume by contradiction that $D$ is not a kernel-perfect digraph, then $D$ contains an induced subdigraph $H$ such that $H$ is a critical kernel imperfect digraph. Let be $f_0 = xy \in A(H)$. By Theorem 24), there exists $C$; a $(C^0_x, D_2)$-normal, directed cycle of odd length passing by $f_0$ and without $V(C)^0_x$-pseudodiagonals. Therefore $C$ is not symmetric. Then $C$ has a pseudodiagonal $f_c = uv$ in $H$, such that every weakly $D_2-$alternated directed cycle $\gamma$ of odd length passing by $f_c$, satisfies $\gamma = \gamma^{(1)}$. From Theorem 24, there exists a $(C^0_u, D_2)$-normal directed cycle $C$, of odd length containing $f$ and having no $(V(C))^{0}_u$-pseudodiagonals. We have proved $C = C^{(1)}$, thus there exists $w \in P(C)$ such that $u \in C^1_w$. Hence $w \in P(C) \cap C^0_u$. Contradicting that $C$ has no $V(C)(C^0_u)$-pseudodiagonals. □

Proposition 39 Denote by $F_{o.a}(D)$ the set of arcs of $D$ contained in no weakly $D_2-$alternated, directed cycle of odd length. Let $D$ be a finite digraph that satisfies $P(\alpha_{D_1}, \preceq)$, then:

$D$ is kernel-perfect if and only if every induced subdigraph $H$ of $D$, such that $A(H) \cap F_{o.a}(D) = \emptyset$ is kernel-perfect.

Proof. This is a direct consequence of Theorem 33 □

References


[773] Gómez Gutiérrez, V. and López de Medrano, S. *Stably parallelizable manifolds are complete intersections of quadrics.*


(2005):


Román, L. Residuated semigroups and the algebraic foundations of quantum mechanics. 8 p.


Román, L. Residuated semigroups and the algebraic foundations of quantum mechanics. 8 p.


(2006):


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