Monochromatic paths and monochromatic sets of arcs in quasi-transitive digraphs

H. Galeana-Sánchez¹, R. Rojas-Monroy² and B. Zavala¹

¹ Instituto de Matemáticas
Universidad Nacional Autónoma de México
Ciudad Universitaria, México, D.F. 04510
México

² Facultad de Ciencias
Universidad Autónoma del Estado de México
Instituto Literario, Centro 50000, Toluca, Edo. de México
México

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Abstract

Let $D$ be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of $D$, respectively. We call the digraph $D$ an $m$-coloured digraph if the arcs of $D$ are coloured with $m$ colours. A directed path is called monochromatic if all of its arcs are coloured alike. A set $N$ of vertices of $D$ is called a kernel by monochromatic paths if for every pair of vertices there is no monochromatic path and if for every vertex $v$ not in $N$ there is a monochromatic path from $v$ to some vertex in $N$. We prove, in Theorem 3.2, that if $D$ is an $m$-coloured quasi-transitive digraph such that for every vertex $u$ of $D$ the set of arcs that have $u$ as initial end point is monochromatic and $D$ contains no $C_3$, then $D$ has a kernel by monochromatic paths.

Keywords: $m$-coloured quasi-transitive digraph, kernel by monochromatic paths, $\gamma$-cycle.

1 Introduction.

For general concepts we refer the reader to [3]. A kernel $N$ of a digraph $D$ is an independent set of vertices of $D$ such that for every $w \in V(D) \setminus N$ there exists an arc from $w$ to $N$. A digraph $D$ is called kernel perfect digraph when every induced subdigraph of $D$ has a kernel. We call the digraph $D$ an $m$-coloured digraph if the arcs of $D$ are coloured with $m$ colours. A path is called monochromatic if all of its arcs are colored alike. If $C$ is a path of $D$ we denote its length
by \(\ell(C)\). A set \(N\) of vertices of \(D\) is called a kernel by monochromatic paths if for every pair of vertices of \(N\) there is no monochromatic path between them and for every vertex \(v\) not in \(N\) there is a monochromatic path from \(v\) to some vertex in \(N\). The closure of \(D\), denoted \(C(D)\), is the \(m\)-coloured digraph defined as follows: \(V(C(D)) = V(D)\), \(A(C(D)) = A(D) \cup \{(u, v) \text{ with color } i \mid \text{there exists a } uv\text{-monochromatic path of colour } i \text{ contained in } D\}\). Notice that for any digraph \(D, C(C(D)) = C(D)\). The problem of the existence of a kernel in a given digraph has been studied by several authors in particular Richardson [16], [17]; Duchet and Meyniel [6]; Duchet [4], [5]; Galeana-Sánchez and V. Neumann-Lara [9], [10]. The concept of kernel by monochromatic paths is a generalization of the concept of kernel and it was introduced by Galeana-Sánchez [7]. In that work she obtained some sufficient conditions for the existence of a kernel by monochromatic paths in an \(m\)-coloured tournament. More information about \(m\)-coloured digraphs can be found in [7], [8], [18].

A digraph \(D\) is called quasi-transitive if \((u, v) \in A(D)\) and \((v, w) \in A(D)\) implies \((u, w) \in A(D)\) or \((w, u) \in A(D)\). The concept of quasi-transitive digraph was introduced by Ghouilá-Houri [14] and has been studied by several authors for example Bang-Jensen and Huang [1], [2]. Ghouilá-Houri [14] proved that an undirected graph can be oriented as a quasi-transitive digraph if and only if it can be oriented as a transitive digraph, these graphs are namely comparability graphs. More information about comparability graphs can be found in [13], [15].

In [12] H.Galena-Sánchez and R. Rojas-Monroy proved that if \(D\) is a digraph such that \(D = D_1 \cup D_2\), where \(D_i\) is a quasi-transitive digraph which contains no asymmetrical infinite outward path (in \(D_i\)) for \(i \in \{1, 2\}\); and every directed cycle of length 3 contained in \(D\) has at least two symmetrical arcs, then \(D\) has a kernel.

For a vertex \(u\) in an \(m\)-coloured digraph \(D\) we denote by \(A^+(u)\) the set of arcs that have \(u\) as initial end point. And we denote by \(C_3\) the directed cycle of length 3 whose arcs are coloured with three distinct colours.

In this paper is proved that if \(D\) is an \(m\)-coloured quasi-transitive digraph such that for every vertex \(u\) of \(D\), \(A^+(u)\) is monochromatic (all of its elements have the same colour) and \(D\) contains no \(C_3\), then \(D\) has a kernel by monochromatic paths.

We will need the following results.

**Theorem 1.1** ([7]). \(D\) has a kernel by monochromatic paths if and only if \(C(D)\) has a kernel.

**Theorem 1.2** (Duchet [4]). If \(D\) is a digraph such that every directed cycle has at least one symmetrical arc, then \(D\) is a kernel-perfect digraph.

## 2 Monochromatic paths.

We will establish some previous lemmas in order to prove the main theorem.

**Lemma 2.1** Let \(D\) be an \(m\)-coloured quasi-transitive digraph such that for every \(u \in V(D)\), \(A^+(u)\) is monochromatic and let \(T = (u = u_0, u_1, ..., u_n = v)\) be a uv-monochromatic path of minimum length contained in \(D\). Then \((u_i, u_j) \notin A(D)\) for every \(i, j \in \{0, ..., n\}\) with \(j > i + 1\). In particular, for every \(i \in \{0, ..., n - 2\}\), \((u_{i+2}, u_i) \in A(D)\).
Lemma 2.2 Let $D$ be an $m$-coloured quasi-transitive digraph such that for every $u \in V(D)$, $A^+(u)$ is monochromatic and let $T = (u = u_0, u_1, \ldots, u_n = v)$ be a $uw$-monochromatic path of minimum length contained in $D$. Then $(u_j, u_i) \in A(D)$ for every $i, j \in \{0, \ldots, n\}$ with $j > i + 1$, unless $|V(T)| = 4$, in which case the arc $(u_3, u_0)$ may be absent.

Proof: If $|V(T)| = 3$, the result follows from Lema 2.1.

When $|V(T)| = 4$, let $T = (u_0, u_1, u_2, u_3)$ be a $u_0u_3$-monochromatic path. By Lemma 2.1 we have $\{(u_3, u_1), (u_2, u_0)\} \subseteq A(D)$, and the arc $(u_3, u_0)$ may be absent.

Now, we proceed by induction on $|V(T)|$.

Suppose that $|V(T)| = 5$. Let $T = (u_0, u_1, u_2, u_3, u_4)$ be a $u_0u_4$-monochromatic path of minimum length, then from Lemma 2.1 and since $D$ is a quasi-transitive digraph we have that $\{(u_4, u_2), (u_3, u_1), (u_2, u_0), (u_4, u_0)\} \subseteq A(D)$. Also, since $\{(u_4, u_0), (u_0, u_1)\} \subseteq A(D)$ and $D$ is a quasi-transitive digraph then $(u_4, u_1) \in A(D)$ or $(u_1, u_4) \in A(D)$. Lemma 2.1 implies that $(u_1, u_4) \notin A(D)$, then $(u_4, u_1) \in A(D)$. Since $\{(u_3, u_4), (u_4, u_0)\} \subseteq A(D)$ and $D$ is a quasi-transitive digraph then $(u_3, u_0) \in A(D)$ or $(u_0, u_3) \in A(D)$. If $(u_0, u_3) \in A(D)$, we have a contradiction with Lemma 2.1. Then $(u_3, u_0) \in A(D)$. We conclude $(u_j, u_i) \in A(D)$ for every $i, j \in \{0, 1, 2, 3, 4\}$ with $j > i + 1$.

Let $T = (u_0, u_1, \ldots, u_n)$ be a monochromatic path of minimum length $n$ with $n \geq 6$.

Let $T_1 = (u_0, u_1, \ldots, u_{n-1})$ and $T_2 = (u_1, \ldots, u_n)$ then $\ell(T_1) \geq 5$ and $\ell(T_2) \geq 5$, by the inductive hypothesis $T_1$ and $T_2$ satisfy that $(u_j, u_i) \in A(D)$ for every $j > i + 1$. Now, we need to prove that $(u_n, u_0) \in A(D)$. Since $(u_2, u_0) \in A(D)$ and $(u_n, u_2) \in A(D)$, and $D$ is a quasi-transitive digraph then $(u_0, u_n) \in A(D)$ or $(u_n, u_0) \in A(D)$. By Lemma 2.1 $(u_0, u_n) \notin A(D)$, thus $(u_n, u_0) \in A(D)$.

Lemma 2.3 Let $D$ be an $m$-coloured quasi-transitive digraph such that for every $u \in V(D)$, $A^+(u)$ is monochromatic. If there exists a $uw$-monochromatic path in $D$ and $(v, u) \notin A(D)$, then one and only one of the following conditions is satisfied:

1. $(u, v) \in A(D)$.
2. $(u, v) \notin A(D)$ and there exists a $uw$-monochromatic path of length 3, $(u = u_0, u_1, u_2, u_3 = v)$ such that $\{(u_2, u_0), (u_3, u_1)\} \subseteq A(D)$. Moreover, there exists no path of length 2 between $u$ and $v$.

Proof: Let $T$ be a $uw$-monochromatic path of minimum length. Clearly the Lemma holds when $\ell(T)$ is 1. So, assume that $\ell(T)$ is at least 2.

If $\ell(T) \geq 4$, it follows from Lemma 2.2 that $(v, u) \in A(D)$, contradicting the hypothesis. Hence $\ell(T) \leq 3$. When $\ell(T) = 3$, let $T = (u = u_0, u_1, u_2, u_3 = v)$, and Lemma 2.1 implies that $\{(u_2, u_0), (u_3, u_1)\} \subseteq A(D)$.

Now, if $T'$ is a path of length 2 from $u$ to $v$ or from $v$ to $u$, since $D$ is a quasi-transitive digraph then $(u, v) \in A(D)$ or $(v, u) \in A(D)$. The hypothesis implies that $(v, u) \notin A(D)$, then $(u, v) \in A(D)$ contradicting the choice of $T$ and assumption $\ell(T) \geq 2$. We conclude that there is no path of length 2 between $u$ and $v$.
3 The main result.

The following definition is a generalization of the concept of cycle.

**Definition 3.1** A $\gamma$-cycle in $D$ is a sequence of vertices of $D$, $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ such that for every $i \in \{0, 1, \ldots, n\}$:

1. There exists a $u_iu_{i+1}$-monochromatic path in $D$, and
2. There is no $u_{i+1}u_i$-monochromatic path in $D$ (notation mod $(n + 1)$).

$n + 1$ is the length of $\gamma$ and will be denoted it by $\ell(\gamma)$.

The following is the main result of this section.

**Theorem 3.2** Let $D$ be an $m$-coloured quasi-transitive digraph such that for every $u \in V(D)$, $A^+(u)$ is monochromatic, and $D$ has no $C_3$. Then $D$ contains no $\gamma$-cycle.

**Proof:** We will proceed by contradiction. Suppose that $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ is a $\gamma$-cycle in $D$ of minimum length. The definition of $\gamma$-cycle and Lemma 2.3 implies that for each $i \in \{0, 1, \ldots, n\}$ only one of the following conditions is satisfied:

i. $(u_i, u_{i+1}) \in A(D)$,

ii. $(u_i, u_{i+1}) \notin A(D)$, there is a $u_iu_{i+1}$-monochromatic path of length 3 and there is no monochromatic path of length 2 between $u$ and $v$.

Then for each $i \in \{0, 1, \ldots, n\}$ we can take $T_i$ a $u_iu_{i+1}$-path as follows:

$T_i = \begin{cases} (u_i, u_{i+1}) \text{ when } (u_i, u_{i+1}) \in A(D) \text{ and} \\ T_i \text{ a } u_iu_{i+1}\text{-monochromatic path of length } 3 \text{ when } (u_i, u_{i+1}) \notin A(D). \end{cases}$

We have the following assertions.

1. $\ell(\gamma) \geq 3$. It is straightforward.

2. There exists $i \in \{0, 1, \ldots, n\}$ such that $T_i$ and $T_{i+1}$ have different colours.

   Suppose, for a contradiction that for each $i \in \{0, \ldots, n\}$, $T_i$ is coloured $a$. Then $T_0 \cup T_1 \cup \cdots \cup T_{n-1}$ is a $u_0u_n$-monochromatic walk which contains a $u_0u_n$-monochromatic path contradicting the definition of $\gamma$-cycle.

   In view of assertion 2, we may suppose w.l.o.g. that $T_0$ is coloured 1 and $T_1$ is coloured 2.

3. There is no $u_2u_0$-monochromatic path in $D$.

   We proceed by contradiction. Suppose that $T = (u_2 = x_0, x_1, \ldots, x_m = u_0)$ is a $u_2u_0$-monochromatic path in $D$. Then we have the following assertions.
3.1. \( T \) is coloured 3, with \( 3 \notin \{1, 2\} \).

If \( T \) is coloured 1 then \( T \cup T_0 \) contains a \( u_2u_1 \)-monochromatic path. And if \( T \) is coloured 2 then \( T_1 \cup T \) contains a \( u_1u_0 \)-monochromatic path. In any case we obtain a contradiction with the definition of \( \gamma \)-cycle.

3.2. \( \ell(T_0) = 3 \).

Suppose, for a contradiction that \( \ell(T_0) \neq 3 \). The definition of \( T_0 \) implies that \( \ell(T_0) = 1 \). Hence \( (u_0, u_1) \in A(D) \). Since \( (x_{m-1}, u_0) \in A(D) \) and \( D \) is a quasi-transitive digraph then \( (x_{m-1}, u_1) \in A(D) \) or \( (u_1, x_{m-1}) \in A(D) \).

If \( (u_1, x_{m-1}) \in A(D) \) then it is coloured 2 because \( A^+(u_1) \) is monochromatic and \( T_1 \) is coloured 2. So, we may assume that \( (u_1, x_{m-1}, u_0) \) is a \( C_3 \), a contradiction.

If \( (x_{m-1}, u_1) \in A(D) \) then it is coloured 3, \( A^+(x_{m-1}) \) is monochromatic and \( (x_{m-1}, x_m = u_0) \) is coloured 3. So we obtain that \( (u_2 = x_0, \ldots, x_m, u_1) \) is a \( u_2u_1 \)-monochromatic path, contradicting the definition of \( \gamma \)-cycle.

3.3. \( (u_0, u_1) \notin A(D), \ (u_1, u_0) \notin A(D) \) and there is no directed path of length two between \( u_0 \) and \( u_1 \).

It follows from assertion 3.2, the definition of \( \gamma \)-cycle and Lemma 2.3.

3.4. \( \ell(T_1) = 3 \).

Suppose, for a contradiction that \( \ell(T_1) \neq 3 \), then the definition of \( T_1 \) implies that \( \ell(T_1) = 1 \), that is, \( T_1 = (u_1, u_2) \). We analyze the different possibilities for \( \ell(T) \).

If \( \ell(T) = 1 \), since \( \{(u_1, u_2), (u_2, u_0)\} \subseteq A(D) \) and \( D \) is a quasi-transitive digraph then \( (u_1, u_2) \in A(D) \) or \( (u_0, u_1) \in A(D) \) contradicting assertion 3.3.

If \( \ell(T) \geq 2 \), since \( \{(u_1, u_2), (u_2, x_1)\} \subseteq A(D) \) and \( D \) is a quasi-transitive digraph then \( (x_1, u_1) \in A(D) \) or \( (u_1, x_1) \in A(D) \). Suppose that \( (x_1, u_1) \in A(D) \) then \( (x_1, u_1) \) is coloured 3 \( (A^+(x_1) \) is coloured 3). Hence \( (u_2, x_1, u_1) \) is a \( u_2u_1 \)-monochromatic path contradicting the definition of \( \gamma \)-cycle.

So, we may assume that \( (u_1, x_1) \in A(D) \). Then \( \{(u_1, x_1), (x_1, x_2)\} \subseteq A(D) \), since \( D \) is a quasi-transitive digraph we have \( (x_2, u_1) \in A(D) \) or \( (u_1, x_2) \in A(D) \). If \( (x_2, u_1) \in A(D) \) then it is coloured 3 \( (A^+(u_2) \) is coloured 3), so we obtain that \( (u_2, x_1, x_2, u_1) \) is a \( u_2u_1 \)-monochromatic path, a contradiction with the definition of \( \gamma \)-cycle. Hence \( (u_1, x_2) \in A(D) \). We may continue in that way and we obtain that for each \( i \in \{1, \ldots, m-1\} \), \( (u_1, x_i) \in A(D) \), in particular \( (u_1, x_{m-1}) \in A(D) \), then \( (u_1, x_{m-1}, u_0) \) is a \( u_1u_0 \)-path of length 2, contradicting assertion 3.3.

We conclude that \( \ell(T_1) = 3 \).

Now, we may assume that for \( i \in \{0, 1\} \) \( T_i = (u_i, v_i, w_i, u_{i+1}) \) and by Lemma 2.3 \( \{(u_{i+1}, v_i), (w_i, u_i)\} \subseteq A(D) \).

3.5. \( (u_0, u_2) \notin A(D) \).

Suppose, for a contradiction that \( (u_0, u_2) \in A(D) \), then \( \{(u_0, u_2), (u_2, v_1)\} \subseteq A(D) \). Since \( D \) is a quasi-transitive digraph we obtain \( (u_0, v_1) \in A(D) \) or \( (v_1, u_0) \in A(D) \).
If \((v_1, u_0) \in A(D)\) then \((u_1, v_1, u_0)\) is a \(u_1u_0\)-monochromatic path, \((A^+(v_1)\) is coloured 2) contradicting the definition of \(\gamma\)-cycle. Hence \((u_0, v_1) \in A(D)\).

Now, since \(\{(u_0, v_1), (v_1, w_1)\} \subseteq A(D)\) and \(D\) is a quasi-transitive digraph it follows \((u_0, w_1) \in A(D)\) or \((w_1, u_0) \in A(D)\).

If \((u_1, v_0) \in A(D)\) then it is coloured 2 and we obtain that \((u_1, v_1, w_1, u_0)\) is a \(u_1u_0\)-monochromatic path contradicting the definition of \(\gamma\)-cycle. If \((w_0, u_1) \in A(D)\) then \((u_0, w_1, u_1)\) is a \(u_0u_1\)-path of length two, contradicting assertion 3.3. We conclude \((u_0, u_2) \notin A(D)\).

3.6. \((w_2, u_0) \notin A(D)\).

Suppose, for a contradiction that \((u_2, u_0) \in A(D)\). Hence \(\{(w_1, u_2), (u_2, u_0)\} \subseteq A(D)\). Since \(D\) is a quasi-transitive digraph then \((w_1, u_0) \in A(D)\) or \((u_0, w_1) \in A(D)\). If \((w_1, u_0) \in A(D)\) then it is coloured 2, so we obtain that \((u_1, v_1, w_0)\) is a \(u_1u_0\)-monochromatic path, contradicting the definition of \(\gamma\)-cycle. If \((u_0, w_1) \in A(D)\) then it is coloured 1 and we have that \((u_0, w_1, u_2, u_0)\) is a \(C_3\), contradicting the hypothesis.

3.7. There is a \(u_2u_0\)-monochromatic path, \(T\), such that \(\ell(T') = 3\) and \(T'\) is coloured 3.

From our initial assumption there is a \(u_2u_0\)-monochromatic path and \((u_0, u_2) \notin A(D)\).

It follows from Lemma 2.3 that there is a \(u_2u_0\)-monochromatic path, \(T'\), such that \(\ell(T') = 1\) or \(\ell(T') = 3\). Assertion 3.6 implies that \(\ell(T') = 3\). Since \(T\) is coloured 3 then \(A^+(u_2)\) is coloured 3, so we obtain that \(T'\) is coloured 3.

Let \(T' = (u_2, v'_2, w'_2, u_0)\), from the Lemma 2.1 we have that \(\{(u_0, v'_2), (w'_2, u_2)\} \subseteq A(D)\).

3.8. \((w_1, v'_2) \in A(D)\).

Since \(\{(w_1, u_2), (u_2, v'_2)\} \subseteq A(D)\) and \(D\) is a quasi-transitive digraph then \((w_1, v'_2) \in A(D)\) or \((v'_2, w_1) \in A(D)\). If \((v'_2, w_1) \in A(D)\) we obtain \(\{(u_0, v'_2), (v'_2, w_1)\} \subseteq A(D)\).

It follows from the fact that \(D\) is a quasi-transitive digraph that \((u_0, w_1) \in A(D)\) or \((w_1, u_0) \in A(D)\). In the case of \((u_0, w_1) \in A(D)\) we obtain \((u_0, w_1, u_2)\) is a \(u_0u_2\)-path of length 2, since \(D\) is a quasi-transitive digraph then \((u_0, u_2) \in A(D)\) or \((u_2, u_0) \in A(D)\), contradicting assertions 3.5 and 3.6. When \((w_1, u_0) \in A(D)\) then it is coloured 2 and \((u_1, v_1, w_1, u_0)\) is a \(u_1u_0\)-monochromatic path, contradicting the definition of \(\gamma\)-cycle. We conclude \((w_1, v'_2) \in A(D)\).

3.9. \((w'_2, w_1) \in A(D)\).

Since \(\{(w_1, v'_2), (w'_2, w_1)\} \subseteq A(D)\) and \(D\) is a quasi-transitive digraph then \((w_1, w'_2) \in A(D)\) or \((w'_2, w_1) \in A(D)\). Suppose that \((w_1, w'_2) \in A(D)\) then \(\{(w_1, w'_2), (w'_2, u_0)\} \subseteq A(D)\).

It follows from the fact that \(D\) is a quasi-transitive digraph that \((w_1, u_0) \in A(D)\) or \((u_0, w_1) \in A(D)\). When \((u_0, w_1) \in A(D)\) we obtain that \((u_0, w_1, w'_2, u_0)\) is a \(C_3\), contradicting our hypothesis. If \((w_1, u_0) \in A(D)\) then it is coloured 2, so we obtain that \((u_1, v_1, w_1, u_0)\) is a \(u_1u_0\)-monochromatic path, contradicting the definition of \(\gamma\)-cycle.

Now, since \(\{(w'_2, w_1), (w_1, u_1)\} \subseteq A(D)\) and \(D\) is a quasi-transitive digraph then \((w'_2, u_1) \in A(D)\) or \((u_1, w'_2) \in A(D)\). When \((w'_2, u_1) \in A(D)\) then it is coloured 3, so we have
that \((u_2, v'_2, w'_2, u_1)\) is a \(u_2u_1\)-monochromatic path, contradicting the definition of \(\gamma\)-cycle. If \((u_1, w'_2) \in A(D)\) then \((u_1, w'_2, u_0)\) is a \(u_1u_0\)-path, contradicting assertion 3.3. We conclude from assertions 3.1-3.9 that there is no \(u_2u_0\)-monochromatic path in \(D\).

4. \(\ell(\gamma) \geq 4\). It follows from assertion 3 and the definition of \(\gamma\)-cycle.

5. There is no \(u_0u_2\)-monochromatic path in \(D\).

Suppose, for a contradiction that there is a \(u_0u_2\)-monochromatic path in \(D\). Since there is no \(u_2u_0\)-monochromatic path in \(D\) (assertion 3), we obtain that \((u_0, u_2, u_3, \ldots u_n, u_0)\) is a \(\gamma\)-cycle of length lesser than \(\ell(\gamma)\), contradicting the choice of \(\gamma\).

6. There is no monochromatic path of length 2 between \(u_0\) and \(u_2\). In particular \((u_0, u_2) \not\in A(D)\) and \((u_2, u_0) \not\in A(D)\). Moreover, there is no path of length 2 between \(u_0\) and \(u_2\).

It follows from assertions 3 and 5 and from the fact that \(D\) is a quasi-transitive digraph.

Now we analyze the four possible cases over \(\ell(T_0)\) and \(\ell(T_1)\).

Case a. \(\ell(T_0) = \ell(T_1) = 1\).

In this case \(\{(u_0, u_1), (u_1, u_2)\} \subseteq A(D)\) and \((u_0, u_1, u_2)\) is a \(u_0u_2\)-path of length 2, a contradiction.

Case b. \(\ell(T_0) = 1, \ \ell(T_1) = 3\).

We have the following affirmations:

b.1. \((u_0, v_1) \in A(D)\).

Since \(\{(u_0, u_1), (u_1, v_1)\} \subseteq A(D)\) and \(D\) is a quasi-transitive digraph, then \((u_0, v_1) \in A(D)\) or \((v_1, u_0) \in A(D)\).

Suppose, for a contradiction that \((v_1, u_0) \in A(D)\), then it is coloured 2, so \((u_1, v_1, u_0)\) is a \(u_1u_0\)-monochromatic path, contradicting the definition of \(\gamma\)-cycle. Hence \((u_0, v_1) \in A(D)\).

b.2. \((u_0, w_1) \in A(D)\).

\(D\) is a quasi-transitive digraph, and \(\{(u_0, v_1), (v_1, w_1)\} \subseteq A(D)\) then \((u_0, w_1) \in A(D)\) or \((w_1, u_0) \in A(D)\).

We suppose that \((w_1, u_0) \in A(D)\), then it is coloured 2 because \(A^+(w_1)\) is coloured 2. So, \((u_1, v_1, w_1, u_0)\) is a \(u_1u_0\)-monochromatic path, contradicting the definition of \(\gamma\)-cycle. Hence \((u_0, w_1) \in A(D)\).

We conclude that \((u_0, w_1, u_2)\) is a \(u_0u_2\)-path of length 2, contradicting 6.

Case c. \(\ell(T_0) = 3, \ \ell(T_1) = 1\).

Since \(\{(w_0, u_1), (u_1, u_2)\} \subseteq A(D)\) and \(D\) is a quasi-transitive digraph then \((w_0, u_2) \in A(D)\) or \((w_2, w_0) \in A(D)\). If \((w_0, u_2) \in A(D)\) then it is coloured 1 (\(A^+(w_0)\) is coloured 1), so we obtain that \((u_0, v_0, w_0, u_2)\) is a \(u_0u_2\)-monochromatic path, contradicting 5. If \((u_2, w_0) \in A(D)\) then \((u_2, u_0, w_0)\) is a \(u_2u_0\)-path of length 2, contradicting 6, (remember \((w_0, u_0) \in A(D)\) by Lemma 2.3).
Case d.  $\ell(T_0) = \ell(T_1) = 3$.

**d.1.** $(w_0, v_1) \in A(D)$.
Since $\{(w_0, u_1), (u_1, v_1)\} \subseteq A(D)$ and $D$ is a quasi-transitive digraph then $(w_0, v_1) \in A(D)$ or $(v_1, w_0) \in A(D)$. If $(v_1, w_0) \in A(D)$ we have $\{(v_1, w_0), (w_0, u_0)\} \subseteq A(D)$ then $(v_1, u_0) \in A(D)$ or $(u_0, v_1) \in A(D)$. If $(v_1, u_0) \in A(D)$ then it is coloured 2 ($A^+(v_1)$ is coloured 2) and $(u_1, v_1, u_0)$ is a $u_1u_0$-monochromatic path, contradicting the definition of $\gamma$-cycle.
Now, suppose that $(v_1, w_0) \in A(D)$ and $D$ is a quasi-transitive digraph we have $(u_0, w_1) \in A(D)$ or $(v_1, w_0) \in A(D)$. If $(v_1, u_0) \in A(D)$ then it is coloured 2 ($A^+(v_1)$ is coloured 2) and $(u_1, v_1, w_0)$ is a $u_1u_0$-monochromatic path, contradicting the definition of $\gamma$-cycle.

**d.2.** $(w_1, w_0) \in A(D)$.
Since $\{(w_0, v_1), (v_1, w_1)\} \subseteq A(D)$ and $D$ is a quasi-transitive digraph then $(w_1, w_0) \in A(D)$ or $(w_0, v_1) \in A(D)$. Assume, for a contradiction that $(w_0, v_1) \in A(D)$. Then $\{(w_0, v_1), (v_1, w_2)\} \subseteq A(D)$, since $D$ is a quasi-transitive digraph then $(w_0, w_2) \in A(D)$ or $(w_0, u_0) \in A(D)$. If $(w_0, u_0) \in A(D)$ then it is coloured 1, thus $(u_0, v_0, w_0, w_2)$ is a $u_0u_2$-monochromatic path, contradicting 5. If $(u_0, w_2) \in A(D)$ then $(w_2, w_0, u_0)$ is a $u_2u_0$-path of length 2, a contradiction with 6.

Now, $\{(w_1, w_0), (v_1, w_0)\} \subseteq A(D)$ and $D$ is a quasi-transitive digraph then $(u_0, v_1) \in A(D)$ or $(w_0, v_1) \in A(D)$. If $(u_0, v_1) \in A(D)$ then $(u_0, v_1, w_2)$ is a $u_0u_2$-path of length 2, contradicting 6. If $(w_1, u_0) \in A(D)$ then it is coloured 2 ($A^+(w_1)$ is coloured 2) and $(u_1, v_1, u_0)$ is a $u_1u_0$-monochromatic path, a contradiction with the definition of $\gamma$-cycle.

Thus $D$ contains no $\gamma$-cycle.

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**Theorem 3.3** Let $D$ be an $m$-coloured quasi-transitive digraph such that for every $u \in V(D)$, $A^+(u)$ is monochromatic, and $D$ has not $C_3$. Then $\mathcal{E}(D)$ is a kernel-perfect digraph.

**Proof:** The proof is straightforward from Theorems 1.2 and 3.2.

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**Corollary 3.4** Let $D$ be an $m$-coloured quasi-transitive digraph such that for every $u \in V(D)$, $A^+(u)$ is monochromatic, and $D$ has not $C_3$. Then $D$ has a kernel by monochromatic path.

**Proof:** From Theorems 1.1 and 3.3.

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**Remark 3.1** The condition $A^+(u)$ monochromatic for every $u \in V(D)$ can not be dropped in the Theorem 3.3. The following digraph $D$ is a counterexample, see figure 1. $D$ no contains $C_3$ and $D$ has no kernel by monochromatic paths. See [11].

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Información o pedidos:
Leonardo Espinosa
Teléfono: 56-22-44-96

Depto. de Publicaciones
FAX: 55-50-13-42 y 56-16-03-48

Instituto de Matemáticas, UNAM
e-mail: leo@matem.unam.mx

Circuito Exterior
web: http://www.matem.unam.mx

Ciudad Universitaria
http://www.smm.org.mx

04510 México, D.F. MÉXICO