(k, l)-kernels in quasi-transitive digraphs*

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Abstract

Let \( D = (V, A) \) be a directed graph (digraph) without loops nor multiple arcs. A set of vertices \( S \) of a digraph \( D \) is a \((k, l)\)-kernel of \( D \) if and only if for any two vertices \( u, v \) in \( S \), \( d(u, v) \geq k \) and for any vertex \( u \) in \( V \setminus S \) there exists \( v \) in \( S \) such that \( d(u, v) \leq l \).

A digraph \( D \) is called quasi-transitive if and only if for any distinct vertices \( u, v, w \) of \( D \) such that \( u \rightarrow v \rightarrow w \), then \( u \) and \( w \) are adjacent vertices in \( D \).

In this paper, we characterize the \((2, 1)\)-kernels (usually knowns as kernels simply) in quasi-transitive digraphs. We prove also that every quasi-transitive digraph possesses a \((3, 2)\)-kernel and we give how to get it.

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1 Preliminaries

For general concepts we refer the reader to [2]. In this paper, \( D = (V(D), A(D)) \) denotes a directed graph (digraph) without loops nor multiple arcs. For each vertex \( u \) in \( D \), \( N^+(u) \) (respectively \( N^-(u) \)) denotes the ex-neighbourhood (resp. in-neighbourhood) of \( u \). \( d^+(u) = |N^+(u)| \) (resp. \( d^-(u) = |N^-(u)| \)). We denote an arc \((u, v)\) in \( A(D) \) by \( u \rightarrow v \). Two distinct vertices \( u \) and \( v \) are adjacent if and only if \( u \rightarrow v \) or \( v \rightarrow u \). If \( S \) is a set of vertices of \( D \), we denote by \( D[S] \) the induced subdigraph by \( S \) in \( D \). All our paths are directed. A \((u, v)\)-path in a digraph \( D \) is a path which initial vertex is \( u \) and which terminal vertex is \( v \). The distance between vertices \( u \) and \( v \) in

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$D$ is the length of the minimum $(u, v)$-path in $D$. If $u$ is a vertex of $D$ and $S$ is a set of vertices of $D$, the distance between $u$ and $S$ is the minimum of the distances between $u$ and each vertex in $S$, analogously for $d(S, u)$. A digraph $D$ is strong if and only if for any vertices $u$ and $v$ in $D$ there exists a $(u, v)$-path in $D$. The components of $D$ are the maximal strong subdigraphs of $D$. A component $D'$ of $D$ is terminal if and only if there is no vertex $u$ not in $D'$ such that $d(D', u) \leq 1$.

A digraph $D$ is quasi-transitive if and only if for any distinct three vertices $u$, $v$ and $w$ in $D$ such that $u \rightarrow v \rightarrow w$, then $u$ and $w$ are adjacent vertices. The quasi-transitive digraphs were introduced by A. Ghouilà-Houri in [9]. They are a well-known class of digraphs that has been widely studied in several papers from different perspectives, as in [1, 8, 11, 10, 15].

A set of vertices $S$ is a kernel of $D$ if and only if $S$ is independent, i.e. there is no pair of vertices adjacent in $S$, and absorbent, i.e. any vertex $u$ in $V(D) \setminus S$ has an ex-neighbour in $S$. The notion of kernel was introduced by J. von Neumann and O. Morgernstern in [13] and has been widely studied in many papers such [3, 4, 6, 7, 9, 14].

As a generalization of the notion of kernel, in [12] M. Kwaśnik defined the $(k, l)$-kernel as a set $S$ of vertices of $D$ such that:

(i) $S$ is $k$-independent, i.e., for any two distinct vertices $u$ and $v$ in $S$, $d(u, v) \leq k$, and

(ii) $S$ is $l$-absorbent in $D$, i.e., for any vertex $u$ in $V(D) \setminus S$, there exists $v$ in $S$ such that $d(u, v) \leq l$.

A kernel is a $(2, 1)$-kernel. $(2, 2)$-kernels are known as quasi-kernels.

## 2 $(k, l)$-kernels in quasi-transitive digraphs

In [1], J. Bang-Jensen and J. Huang characterize the quasi-transitive digraphs. They also prove the followings:

**Proposition 2.1** (Bang-Jensen and Huang [1]). Let $D$ be a quasi-transitive digraph. Suppose that $P = (x_1, x_2, \ldots, x_k)$ is a minimal $(x_1, x_k)$-path. Then the subdigraph induced by $V(P)$ is a semicomplete digraph and $x_j \rightarrow x_i$ for every $2 \leq i + 1 < j \leq k$, unless $k = 4$, in which case the arc between $x_1$ and $x_k$ may be absent.

**Corollary 2.2** (Bang-Jensen and Huang [1]). If a quasi-transitive digraph $D$ has an $(x, y)$-path but $x$ does not dominate $y$, then either $y \rightarrow x$, or there exist vertices $u, v \in V(D) \setminus \{x, y\}$ such that $x \rightarrow u \rightarrow v \rightarrow y$ and $y \rightarrow u \rightarrow v \rightarrow x$.

Note that if there is directed $(x, y)$-path in a quasi-transitive digraphs, then $x$ and $y$ are in the same component of $D$.

**Corollary 2.3.** Let $D$ be a strong quasi-transitive digraph and $S$ an independent set of vertices of $D$. Then $S$ is 3-independent but $k$-independent for $k \geq 4$. 


Proof. Let \( S \) be as in the hypothesis of the proposition. Take distinct vertices \( u \) and \( v \) in \( S \) and let \( P \) be a directed \((u, v)\)-path. Since there is no \( u \to v \), it follows from Corollary 2.2 that \( v \to u \) or there exist \( x, y \in V(D) \setminus \{u, v\} \) such that \( u \to x \to y \to v \) and \( v \to x \to y \to u \). But \( S \) is independent, therefore \( d_D(u, v) = d_D(v, u) = 3 \).

Therefore, it follows from the previous remark and corollary that in a quasi-transitive digraph there exist only two classes of independent set of vertices, the first one is \(3\)-independent but \( k\)-independent for \( k = 4, 5, \ldots \) and the other one is \( k\)-independent for all \( k = 1, 2, \ldots \).

**Theorem 2.4** (Chvátal and Lovász [5]). Every digraph \( D \) has a quasi-kernel.

**Corollary 2.5.** Every quasi-transitive digraph \( D \) has a \((3, 2)\)-kernel.

**Proof.** It follows from Theorem 2.4 and Corollary 2.3.

Moreover, in [11] S. Heard and J. Huang proved the following theorem:

**Theorem 2.6** (Heard and Huang [11]). Every quasi-transitive digraph with no sink contains a pair of disjoint quasi-kernels.

Where a vertex \( v \) is a sink of \( D \) if and only if \( d^+(v) = 0 \).

**Corollary 2.7.** Every quasi-transitive digraph with no sink contains a pair of disjoint \((3, 2)\)-kernels.

The question remains whether a quasi-transitive digraphs has a kernel or not.

**Theorem 2.8.** Let \( D \) be a strong quasi-transitive digraph. \( D \) has a kernel if and only if \( D \) has a vertex \( v \) such that \( d^-(v) = |V(D)| \).

**Proof.** For the necessary condition, \( v \) is a kernel of \( D \).

We prove the sufficient condition by contrapositive. For any vertex \( v \) in \( D \), assume \( d^-(v) < |V(D)| \). Take \( S \) any independent set of vertices of \( D \). We will prove that \( S \) is not absorbent. If \(|S| = 1\), then \( S \) is not a kernel by our assumption. Thus, suposse \(|S| \geq 2\) and take \( x, y \in S \).

Because Corollary 2.2, there exist vertices \( u \) and \( v \) in \( D \) such that \( x \to u \to v \to wy \) and \( y \to u \to v \to x \). Thus \( d(u, x) = d(u, y) = 2 \). If there exists \( w \in S \) such that \( u \to w \), then we have \( x \to u \) and \( u \to w \). Since \( D \) is a quasi-transitive digraph, it follows that \( x \) and \( w \) are adjacent, a contradiction because we assume \( S \) as a kernel of \( D \).

The following theorem provides a way to construct a \((3, 2)\)-kernel in a quasi-transitive digraph.

**Theorem 2.9.** Let \( D \) be a strong quasi-transitive digraph. If \( S \subseteq N^+(x) \setminus N^-(x) \) is a \((3, 2)\)-kernel of \( D[N^+(x) \setminus N^-(x)] \) for any vertex \( x \) in \( D \), then \( S \) is a \((3, 2)\)-kernel of \( D \).

**Proof.** By the Corollary 2.3, \( S \) is \(3\)-independent in \( D \). We will prove that \( S \) is \(2\)-absorbent in \( D \). Take any \( y \in V(D) \). If \( d(y, x) \leq 1 \) or \( d(x, y) \leq 1 \), then \( d(y, S) \leq 2 \) by the definition of \( S \). Otherwise, there exist \( u, v \) in \( V \setminus \{x, y\} \) such that \( x \to u \to v \) and \( y \to u \to v \). There are no \( u \to x \), \( x \to v \), \( u \to y \), \( y \to v \) because \( D \) is quasi-transitive and \( \{x, y\} \) is an independent set of vertices. Therefore \( u \in N^+(x) \setminus N^-(x) \) and \( d(u, S) \leq 2 \) in \( D[N^+(x) \setminus N^-(x)] \). Thus there exists \( w \in N^+(x) \setminus N^-(x) \) such that \( u \to w \) and \( d(w, S) \leq 1 \). Since there are \( y \to u \) and \( u \to w \), it follows from the quasi-transitivity that \( y \) and \( w \) are adjacent vertices in \( D \). If \( w \to y \), then \( x \) and \( y \) are adjacent because \( x \to w \), a contradiction. Therefore, \( y \to w \) and \( d(y, S) \leq 2 \).
Proposition 2.10. Let $D$ be a quasi-transitive digraph and $D_1, \ldots, D_p$ its terminal components. For any vertex $v$ in $V(D) \setminus \bigcup_{i=1}^{p} V(D_i)$, $d(v, \bigcup_{i=1}^{p} V(D_i)) \leq 1$.

Proof. It follows from Proposition 2.1.

Corollary 2.11. Let $D$ be a quasi-transitive digraph and $D_1, \ldots, D_p$ its terminal components. Let $S_i \subseteq V(D_i)$ be a $(k, l)$-kernel of $D_i$ for each $i = 1, 2, \ldots, p$. Then $\bigcup_{i=1}^{p} S_i$ is a $(k, l)$-kernel of $D$.

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