A conjecture of Neumann-Lara on infinite families of r-dichromatic circulant tournaments

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Abstract

In this paper we exhibit infinite families of vertex critical r-dichromatic circulant tournaments for all \( r \geq 3 \). The existence of these infinite families was conjectured by Neumann-Lara (7), who later proved it for all \( r \geq 3 \) and \( r \neq 7 \). Using different methods we find explicit constructions of these infinite families for all \( r \geq 3 \), including the case when \( r = 7 \), which completes the proof of the conjecture.

Key words: Circulant tournament, dichromatic number, vertex critical.

1 Introduction

Let \( D \) be a digraph and \( V(D) \), \( A(D) \) denote the sets of vertices and arcs of \( D \) respectively. The digraph \( D \) is acyclic if it contains no directed cycles. A subset of \( V(D) \) which induces an acyclic subdigraph of \( D \) will be called acyclic. An acyclic tournament of order \( n \) is called a transitive
tournament and is denoted by $TT_n$. We say that a tournament $T$ is $TT_n$-free if $T$ does not have a
subgraph isomorphic to $TT_n$.

The dichromatic number of a digraph $D$, denoted $dc(D)$, was introduced in (5) as the minimum
number of colors needed to color the vertices of $D$ so that each chromatic class induces an acyclic
subdigraph of $D$. A digraph is called $r$-dichromatic if $dc(D) = r$, and vertex critical $r$-dichromatic
if $dc(D) = r$ and $dc(D \setminus v) < r$ for all $v \in V(D)$.

Let $\mathbb{Z}_{2n+1}$ be the set of integers mod$(2n+1)$ and $J$ a subset of $\mathbb{Z}_{2n+1} \setminus \{0\}$ such that $|\{j, -j\} \cap J| =
1$ for every $j \in \mathbb{Z}_{2n+1} \setminus \{0\}$. The circulant tournament $\overrightarrow{C_{2n+1}}(J)$ is defined as follows:

$V(\overrightarrow{C_{2n+1}}(J)) = \mathbb{Z}_{2n+1}$ and

$A(\overrightarrow{C_{2n+1}}(J)) = \{(i, j) : i, j \in \mathbb{Z}_{2n+1}, j - i \in J\}$. Recall that
the circulant tournaments are vertex transitive.

The vertex critical $r$-dichromatic tournaments have been studied in several papers (7; 9; 10). In
(7) Neumann-Lara conjectured that there exists an infinite family of $r$-dichromatic vertex critical
circulant tournaments for all $r \geq 3$. He proved this conjecture for all $r \geq 3$ and $r \neq 7$ (9), using
composition of tournaments.

In this paper we exhibit explicitly infinite families of vertex critical $r$-dichromatic circulant
tournaments for each $r \geq 3$. The conjecture is then completely settled since our constrution includes
the case $r = 7$.

It is important to note that we do not use compositions of tournaments to construct these infi-
nite families. We generalize the infinite families of 3 and 4 dichromatic vertex critical circulant
tournaments showed in (7; 10). The infinite families constructed in this paper for $r \geq 5$, are not
isomorphic to the ones in (9). Moreover, our infinite families include the special tournaments $ST_7$, $ST_{13}$, the largest circulant tournaments free of transitive subtournaments of order 4 (3) and 5 (11),
and the special tournament $ST_{31}$ (12).

In the second section we construct specific tournaments. Each tournament is encoded by a suitable
matrix which naturally reflects its adjacency structure.

In the third section we prove that for each $n \geq 2$ there exists an infinite family of $(n + 1)$-
dichromatic vertex critical circulant tournaments of order $n (p + 1)+1$ with $p=(n - 1)(s + 1)+1$,
$s \geq 0$ in which the induced graph by the exneighborhood of each vertex is isomorphic to the tour-
nament introduced in the first section. Moreover, we prove that all elements of the family do not
contain transitive subtournaments of order $p + 2$, but they have a descomposition into $n$ disjoint
transitive tournaments of order $p+1$ and one isolated vertex. This fact leads immediatly to a $(n+1)$-
vertex coloring which is critical by vertex transitivity.
For general concepts we refer the reader to (1; 2).

2 The family of $\nabla_{p,s}$-triangles

In this section we construct for all $n \geq 1$ a family of tournaments, denoted by $\nabla_{p,s}$, for $p = (s + 1)(n - 1) + r$, $0 < r \leq s + 1$, and $p \geq s + 2$. For each $s \geq 0$ there is such a tournament in this family. We define the set of vertices $V(\nabla_{p,s})$, denoted by $\nabla_{p,s}$, as the following subset of entries $m_{ij}$ in a $M_{n \times (p+1)}$ matrix,

$$\nabla_{p,s} = \{m_{ij} \in M_{n \times (p+1)} : (i - 1)(s + 1) + j \leq p\}$$

We define the exneighborhood of each vertex $m_{ij} \in \nabla_{p,s}$ as the union of three disjoint sets of vertices in $\nabla_{p,s}$, specifically $N^+(m_{ij}, \nabla_{p,s}) = A_{ij} \cup B_{ij} \cup C_{ij}$, where

$$A_{ij} = \{m_{k+1,j+l} \in M_{n \times (p+1)} : k(s + 1) + l \leq (i - 2)(s + 1), k, l \geq 0\},$$

$$B_{ij} = \{m_{i+k,j+l+1} \in M_{n \times (p+1)} : k(s + 1) + l < (n - i)(s + 1) - j + r, k, l \geq 0\},$$

$$C_{ij} = \{m_{i+k+1,l+1} \in M_{n \times (p+1)} : k(s + 1) + l \leq j - 2, k, l \geq 0\}.$$  

In the next figure the entry $m_{ij}$ is the point $\ast$, the exneighborhood of $\ast$ are the $\bullet$ points partitioned into the sets $A_{ij}$, $B_{ij}$ and $C_{ij}$ (the triangles that appear at the top, to the right and to the left respectively). Note that the inneighborhood of $\ast$ are the $\circ$ points and the points $\cdot$ are not vertices of $\nabla_{p,s}$, and in the last row, only the first $r$ entries of the matrix are vertices of the tournament $\nabla_{p,s}$.

**Lemma 1** The tournaments induced by $A_{ij}$, $B_{ij}$, $C_{ij}$ are isomorphic to

$$\nabla_1+(i-2)(s+1),s, \nabla_{p-(i-1)(s+1)-j,s}, \nabla_{j-1,s}$$

respectively.

**Proof.** Let $\Psi_A : A_{ij} \rightarrow \nabla_1+(i-2)(s+1),s$, $\Psi_B : B_{ij} \rightarrow \nabla_{p-(i-1)(s+1)-j,s}$, $\Psi_C : C_{ij} \rightarrow \nabla_{j-1,s}$, defined as follows:
Figure 1. The tournament $\nabla_{23,2}$. In this example $m_{ij} = m_{3,6}$.

$m_{kl} \in A_{ij}$, then $\Psi_A (m_{kl}) = m_{k,l-j+1}$.
$m_{kl} \in B_{ij}$, then $\Psi_B (m_{kl}) = m_{k-i+1,l-j}$
$m_{kl} \in C_{ij}$, then $\Psi_C (m_{kl}) = m_{k-i,l}$.

Clearly $\Psi_A, \Psi_B, \Psi_C$ are all isomorphisms.

Proposition 1 $\nabla_{p,s}$ is $TT_{p+1}$-free, moreover the maximum order of a transitive subtournament in $\nabla_{p,s}$ is $p$.

Proof. The set $\{m_{1j} \in M_{n \times (p+1)} : 1 \leq j \leq p\}$ is acyclic and has order $p$, then the maximum order of a transitive subtournament in $\nabla_{p,s}$ is at least $p$.

If $1 \leq p \leq s + 1$, then $n = 1$ and $p = r$. Therefore the tournament $\nabla_{p,s}$ is the set

$\{m_{1j} \in M_{1 \times (p+1)} : 1 \leq j \leq p\}$

and $\nabla_{p,s}$ is a transitive tournament of order $p$.

We prove by induction on $p$, for $p \geq s + 2$, that $\nabla_{p,s}$ is $TT_{p+1}$-free.

Now $n \geq 2$ and $p = (s + 1)(n - 1) + r, 0 < r \leq s + 1$.

Fix $s \geq 0$. Let $p = s + 2$, then $\nabla_{p,s}$ is isomorphic to the exneighborhood of 0 of the circulant tournament $\overrightarrow{C}_{2(p+1)+1} (1, 2, \ldots, p, p+2)$ and this exneighborhood is $TT_{p+1}$-free (10).

Suppose that all $\nabla_{k,s}$ is $TT_{k+1}$-free, for $s + 2 < k < p$. 

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Let $TT_q$ be a transitive tournament in $\nabla_{p,s}$. We prove that $q \leq p$.

Let $m_{ij}$ be the source of $TT_q$. Then $N^+(m_{ij})$ is the disjoint union of transitive subtournaments of $A_{ij}, B_{ij}, C_{ij}$. Let $T_U = TT_q[U]$, for $U \in \{A_{ij}, B_{ij}, C_{ij}\}$. So

$$V(TT_q) = \{m_{ij}\} \cup V(T_{A_{ij}}) \cup V(T_{B_{ij}}) \cup V(T_{C_{ij}})$$

and since $T_{A_{ij}}, T_{B_{ij}}, T_{C_{ij}}$ are disjoint, then

$$|V(TT_q)| = 1 + |V(T_{A_{ij}})| + |V(T_{B_{ij}})| + |V(T_{C_{ij}})|.$$

By Lemma 1 and by induction hypothesis, it follows that $|V(T_{A_{ij}})| \leq 1 + (i - 2)(s + 1)$, $|V(T_{B_{ij}})| \leq p - (i - 1)(s + 1) - j$, $|V(T_{C_{ij}})| \leq j - 1$.

Let $i = 1$, then $A_{1,j} = \emptyset$, and

$$|V(TT_q)| \leq 1 + 0 + (p - j) + (j - 1) = p$$

Let $i \geq 2$, then

$$|V(TT_q)| \leq 1 + (1 + (i - 2)(s + 1)) + (p - (i - 1)(s + 1) - j) + (j - 1) = p - s$$

Then the maximum order of a transitive subtournament in $\nabla_{p,s}$ is $p$. □

3 The infinite families of $(n + 1)$-dichromatic vertex critical circulant tournaments.

In this section we construct, for all $n \geq 2$, an infinite family of vertex critical circulant tournaments with dichromatic number $n + 1$, denoted by $\overrightarrow{C}_{n(p+1)+1}(\nabla_{p,s})$, for $p = (s + 1)(n - 1) + 1$. Note that for each $s \geq 0$ there is such a tournament in this family. We prove that $N^+(i)$ induces a tournament isomorphic to $\nabla_{p,s}$ and then we use this fact to conclude the properties of the family $\overrightarrow{C}_{n(p+1)+1}(\nabla_{p,s})$.

Let $p = (n - 1)(s + 1) + 1$, and $\overrightarrow{C}_{n(p+1)+1}(J)$, with
\[ J = \{1, 2, \ldots, p\} \cup \{p + 2, p + 3, \ldots, 2p - s\} \cup \{2p + 3, \ldots, 3p - 2s\} \ldots \cup \{(n - 1)p + n\} \]

Note that \(|\{i, -i\} \cap J| = 1\) for all \(i \in \mathbb{Z}_{n(p+1)+1} \setminus \{0\}\), then \(\overrightarrow{C}_{n(p+1)+1}(J)\) is a circulant tournament.

**Lemma 2** The tournament induced by \(N^+(i)\) is isomorphic to \(\nabla_{p,s}\).

**Proof.** Since \(\overrightarrow{C}_{n(p+1)+1}(J)\) is vertex transitive we may assume that \(i = 0\). Let \(T\) induced by \(N^+(0)\). Note that \(N^+(0) = J\).

Let \(v \in V(T)\), and \(i, j \in \mathbb{N}\) such that \(v = (p + 1)(i - 1) + j\), with \(1 \leq i \leq n\) and \(1 \leq j \leq p\). We define the morphism \(\psi : V(T) \to \nabla_{p,s}\) as follows: \(\psi(v) = m_{ij}\). Clearly \(\psi\) is an isomorphism between the subtournament induced by \(N^+(0)\) and the tournament \(\nabla_{p,s}\). It is important to note that in figure 1 the inneighborhood of 0 are the points \(\cdot\) and the exneighborhood are the points \(\{\circ, \bullet\}\). \(\square\)

From now on we denote the tournament \(\overrightarrow{C}_{n(p+1)+1}(J)\) by \(\overrightarrow{C}_{n(p+1)+1}(\nabla_{p,s})\).

**Theorem 1** The maximum order of a transitive subtournamnet in \(\overrightarrow{C}_{n(p+1)+1}(\nabla_{p,s})\) is \(p + 1\).

**Proof.** The proof follows by the Proposition 1, and the fact that \(\{0, 1, \ldots, p\}\) induces a transitive subtournament of \(\overrightarrow{C}_{n(p+1)+1}(\nabla_{p,s})\). \(\square\)

**Theorem 2** \(\overrightarrow{C}_{n(p+1)+1}(\nabla_{p,s})\) is a \((n + 1)\)-dichromatic vertex critical circulant tournament.

**Proof.** By Theorem 1 the chromatic classes are of order at most \(p + 1\), then

\[
dc(\overrightarrow{C}_{n(p+1)+1}(\nabla_{p,s})) \geq \left\lceil \frac{n (p + 1) + 1}{p + 1} \right\rceil = n + 1
\]
We define the coloring $\varphi : V \left( \overrightarrow{C}_{n(p+1)+1}(\nabla_{p,s}) \right) \to I_{n+1}$ as follows:

$$\varphi (v) = \begin{cases} 
  k, & \text{if } v \in \{k(p+1) + \{1, 2, \ldots, p+1\}, k \in \{0, 1, \ldots, n-1\}\} \\
  n, & \text{if } v = 0
\end{cases}$$

$\varphi$ is a $(n+1)$-coloring without monochromatic cycles, then

$$dc \left( \overrightarrow{C}_{n(p+1)+1}(\nabla_{p,s}) \right) = n+1$$

Moreover $\overrightarrow{C}_{n(p+1)+1}(\nabla_{p,s})$ is vertex critical since it is vertex transitive and $\varphi$ has a class with only one element. $\square$

Note that for $n = 2$ the family $\overrightarrow{C}_{n(p+1)+1}(\nabla_{p,s})$ is isomorphic to the family

$$\overrightarrow{C}_{2m+1}(1, 2, \ldots, m-1, m+1)$$

which is 3-dichromatic.

For $n = 3$ the family $\overrightarrow{C}_{n(p+1)+1}(\nabla_{p,s})$ is isomorphic to the family

$$D_m = \overrightarrow{C}_{6m+1}(1, 2, \ldots, 2m-1, 2m+1, 2m+2, \ldots, 3m, 4m+1)$$

which is 4-dichromatic.

Let $s = 0$, then $p = n$ and we have a family, which has exactly one $(n+1)$-dichromatic circulant tournament for each $n \geq 2$.

**Corollary 1** $\overrightarrow{C}_{n^2+n+1}(\nabla_n)$ is $(n+1)$-dichromatic vertex critical for each $n \geq 2$.

Note that for $n = 2, 3, 5$ we have the special tournaments $ST_7(3), ST_{13}(11)$ and $ST_{31}(12)$.

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